

Collective Excitations in QCD Plasma

Navid Abbasi^{a1}, Davood Allahbakhshi^{a2}, Ali Davody^{b,a3}, Seyed Farid Taghavi^{a4};

^a School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5531, Tehran, Iran

^b Institute of Theoretical Physics, Regensburg University, 93040 Regensburg, Germany

ABSTRACT: We study the long wavelength excitations in rotating QCD fluid coupled to an external magnetic field at finite vector and axial charge densities. We first find the generalization of the both well-known Chiral Magnetic Wave (CMW) and Chiral Vortical Wave (CVW), separately. It turns out that at $\mu_5 = 0$ and in the absence of rotation, there exist two CMWs which propagate in the same and in the opposite directions of the magnetic field with the same velocities. However when $\mu_5 \neq 0$, one of the CMW modes propagates faster than the other and additionally, they do not necessarily propagate in the opposite directions. The similar situation happens for the two CVWs in the rotating fluid at finite axial chemical potential. We then show that in general, when the fluid is either rotating and is coupled to a magnetic field, the CMW and the CVW mix with each other and make the Chiral Magnetic-Vortical Wave (CMVW). The resultant coupled waves have generally different velocities compared to the sum of velocities of the individual waves. We also find another excitation in the QCD plasma; the so-called Chiral Alfvén Wave (CAW), analogue of what was recently found in a chiral fluid with single chirality. We specifically show that in contrast to the latter case, the CAWs in QCD fluid may propagate only when both the vector and axial charge densities are non-vanishing. Furthermore, while the velocity of CAWs in a chiral fluid with single chirality depends on the coefficient of gravitational anomaly, we show that in QCD fluid, it depends on the coefficients of both chiral and gravitational anomalies.

¹Abbasi@ipm.ir

²Allahbakhshi@ipm.ir

³Davody@ipm.ir

⁴s.f.taghavi@ipm.ir

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 1 |
| 2 | Chiral fluid and Review of Results | 3 |
| 2.1 | QCD Fluid | 3 |
| 2.2 | Results Addressed | 5 |
| 2.2.1 | General Results | 5 |
| 2.2.2 | Special Limits | 7 |
| 2.2.3 | Results Associated with Quark Gluon Plasma | 9 |
| 3 | QCD Fluid Coupled to External Magnetic Field | 10 |
| 3.1 | Equations of Motion Linearized | 11 |
| 3.2 | Characteristics and Riemann Invariants | 11 |
| 3.3 | Scalar Sector: Chiral Magnetic Waves | 12 |
| 3.4 | Mixed Scalar-Vector Sector | 15 |
| 4 | Rotating QCD Fluid | 20 |
| 4.1 | Equations of Motion Linearized | 20 |
| 4.2 | Hydro Modes | 21 |
| 4.2.1 | Scalar Sector: Chiral Vortical Waves | 21 |
| 4.2.2 | Scalar-Vector Sector | 23 |
| 5 | Rotating QCD Fluid Coupled to Magnetic Field | 24 |
| 5.1 | Equations of Motion Linearized | 24 |
| 5.2 | Hydro Modes | 25 |
| 5.3 | Hydro Modes in Special Case $\mathbf{k} \parallel \mathbf{B} \parallel \mathbf{\Omega}$ | 30 |
| 5.4 | Heavy Ion Collision | 30 |
| 5.4.1 | Comparison With Kharzeev-Yee Result [28] at Zero Density | 31 |
| 5.4.2 | CVW at Constant Temperature; Comparison With Jiang-Huang-Liao [32] Result | 32 |
| 5.4.3 | Chiral Heat Wave (CHW) | 33 |
| 6 | Conclusion and Outlook | 35 |

Contents

1 Introduction

The phenomenon of chiral (or anomaly induced) transport was firstly studied in the fermionic systems either weakly coupled to the electromagnetic field [1] or in presence of rotation in the system [2]. Due to presence of anomaly in the chiral system, macroscopic currents may be produced along the external magnetic field (Chiral Magnetic Effect, CME) or along the vorticity of the system (Chiral Vortical Effect, CVE). The appearance of such currents is the main feature of the chiral transport theory which has been extensively studied in the literature. For example in the context of kinetic theory, a chiral theory has been derived from the underlying quantum field theory [5, 6] in which, the Berry monopole is responsible for the CME and CVE [6, 7]. Chiral magnetic effect has been also studied numerically via lattice field theory [8–11].

In another direction, after the fluid-gravity duality showed the possibility of presence of the missed vorticity term [3, 4], the issue of chiral transport was taken under study in hydrodynamics. At first sight, the parity breaking terms like magnetic field and the vorticity seem to be in contradiction with the existence of a positive divergence entropy current in fluid dynamics, however the necessity of the second law of thermodynamics makes a relation between these terms of the hydrodynamic currents with the underlying quantum anomalies [12].

In contrast to the coefficients of the dissipative transport, the coefficients of the parity odd terms may be entirely fixed in terms of both the anomaly coefficients and the thermodynamic variables. The anomaly induced transport is in fact a non-dissipative phenomenon and so the associated coefficients are referred to as the so-called non-dissipative transport coefficients [21]. This kind of hydrodynamic transport has been used to effectively describe different phenomena in physics; e.g. in neutron stars or supernova in astrophysics [22–24], in the study of the origin of the magnetic fields in cosmology [25] and in Weyl semi-metals in condensed matter physics [26] ¹.

It has been argued that in a plasma of chiral fermions, e.g. the heavy ion plasma, the combination of the Chiral Separation Effect (CSE)² and CME gives rise to the propagation of a new kind of gapless excitation through the plasma; it is called the Chiral Magnetic Wave (CMW) [28]. CMWs have been exploited to predict the charge asymmetries in the final state of a heavy ion collision [29]. Consistent with the prediction of chiral transport, the charge asymmetries have been actually detected

¹ For a complete and recent review of the chiral phenomena see [27] and references therein.

²CSE refers to the separation of chiral charge along the external magnetic field at finite vector charged density.

in experiments at RHIC and LHC [30, 31]. The similar predictions have been made for the propagation of Chiral Vortical Wave (CVW) in heavy ion plasma in [32].

The CMW found in [28] has been computed by considering the fluctuations of vector and axial charge densities, keeping the local energy and local momentum in the plasma fixed. The situation might be more general if one considered finite background vector and axial chemical potentials and also allowed the energy and momentum to hydrodynamically fluctuate. In this paper we do so and compute the corresponding spectrum of the hydrodynamic excitations in a QCD type plasma coupled to an external magnetic field. Consequently, in addition to the two generalized CMWs, we find another four collective excitations, each of them being a coherent perturbations of all six hydrodynamic variables.

We also repeat the above computations for the case of a rotating QCD type plasma. As a result we find the generalization of CVWs found in [32]. Since our results all have been obtained at arbitrary finite vector and axial chemical potentials in a fluid with general equation of state, it would be interesting to compare them with the associated well-known results obtained for special cases. To this end, we compare our result on CMW and CVW with those found in [28] and [32] respectively. While the previous results on CVW in [32] did not capture the gravitational anomaly, we show that the velocity of CVWs depends on the coefficient of the gravitational anomaly as well as the chiral anomaly coefficient.

As a main part in the paper, we compute the hydrodynamic excitations in a rotating plasma which is simultaneously coupled to an external magnetic field. We find the mixed modes of vortical and magnetic effects and show that, in heavy ion plasma where the vorticity is along the direction of the magnetic field, the velocity of mixed waves is exactly equal to the sum of the velocities of the CMW and CVW. It has to be noted that this result had been previously found in a fluid with only vector and axial currents [37, 38].

Another novel result in this paper is that within a QCD type fluid, Chiral Alfvén Wave (CAW) may propagate as well. The propagation of CAW was first predicted theoretically in a chiral fluid with a single chirality [34, 35]. It has been shown that the linear fluctuations of the vorticity may couple to the magnetic field and produce a wave of momentum perturbations propagating parallel to the magnetic field. This wave, namely CAW, might even propagate at zero density in the single chirality fluid. We will show that for the propagation of the CAW in QCD fluid, it is needed both the vector and axial chemical potentials to be non-zero.

The paper has been organized as follows. We begin with a brief review of the chiral QCD plasma in section 2. In the same section we devote a remarkable part to review of our results and give their associated addresses in the paper. The content of sections 3 and 4 is related to detailed computations of magnetic field and vorticity respectively. In section 5 we consider the general case wherein, both the magnetic

field and the vorticity are present. After computing the modes, we discuss their possible implications to heavy ion plasma. We end with conclusion and mentioning some follow up questions in the section 6.

2 Chiral fluid and Review of Results

2.1 QCD Fluid

We consider the QCD matter in an external magnetic field. In addition to usual electric charge (with vector current $J^\mu = \bar{\psi}\gamma^\mu\psi$), this matter carries the chiral charge (with chiral current $J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$). In presence of a background gauge field A_μ , the dynamical equations for this hot matter are nothing but the following conservation equations:

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= F^{\nu\lambda} J_\lambda \\ \partial_\mu J^\mu &= 0 \\ \partial_\mu J_5^\mu &= \mathcal{C} E_\mu B^\mu\end{aligned}\tag{2.1}$$

where $J^\mu = J_R^\mu + J_L^\mu$ and $J_5^\mu = J_R^\mu - J_L^\mu$ and \mathcal{C} is the chiral anomaly coefficient. In long wavelength regime when the matter is in local equilibrium state, the energy momentum tensor $T^{\mu\nu}$, vector current J^μ and the chiral current J_5^μ may all be effectively expressed in terms of six degrees of freedom: three components of the flowing matter velocity u^μ , energy density ϵ , vector charge density n and axial charge density n_5 . We may also define the electric and magnetic field in the rest frame of this fluid as $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$, $E^\mu = F^{\mu\nu}u_\nu$ [12]. For small deviations from local equilibrium state, each of constitutive relations of the fluid may be given in a derivative expansion:

$$\begin{aligned}T^{\mu\nu} &= (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} + \tau^{\mu\nu} \\ J^\mu &= nu^\mu + \nu^\mu, \\ J_5^\mu &= n_5 u^\mu + \nu_5^\mu\end{aligned}\tag{2.2}$$

with $\tau^{\mu\nu}$, ν^μ and ν_5^μ as the derivative corrections to fluid currents. In the Landau-Lifshitz frame where $u_\mu \tau^{\mu\nu} = 0$, $u_\mu \nu^\mu = 0$ and $u_\mu \nu_5^\mu = 0$ [13], up to first order in derivative expansion we have³

$$\tau^{\mu\nu} = -\eta P^{\mu\alpha} P^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \left(\zeta - \frac{2}{3}\eta\right) P^{\mu\nu} \partial \cdot u\tag{2.3}$$

$$\nu^\mu = -\sigma T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T}\right) + \sigma E^\mu + \xi \omega^\mu + \xi_B B^\mu\tag{2.4}$$

$$\nu_5^\mu = -\sigma_5 T P^{\mu\nu} \partial_\nu \left(\frac{\mu_5}{T}\right) + \sigma_5 E^\mu + \xi_5 \omega^\mu + \xi_{B5} B^\mu.\tag{2.5}$$

³ $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu \partial_\alpha u_\beta$.

The coefficients η , ζ , σ and σ_5 are dissipative transport coefficients. In the following we restrict our study to non-dissipative fluids. So the only relevant coefficients are the anomalous transport coefficients ξ and ξ_B corresponding to CVE and CME [12, 14–18]

$$\begin{aligned}\xi &= \mathcal{C} \left(\mu\mu_5 - \frac{n\mu_5}{3w} (3\mu^2 + \mu_5^2) \right) - \mathcal{D} \frac{n\mu_5}{w} T^2 \\ \xi_5 &= \frac{\mathcal{C}}{2} \left(\mu^2 + \mu_5^2 - \frac{2n_5\mu_5}{3w} (3\mu^2 + \mu_5^2) \right) + \frac{\mathcal{D}}{2} \left(1 - \frac{2n_5\mu_5}{w} \right) T^2 \\ \xi_B &= \mathcal{C} \mu_5 \left(1 - \frac{n\mu}{w} \right) \\ \xi_{5B} &= \mathcal{C} \mu \left(1 - \frac{n_5\mu_5}{w} \right)\end{aligned}\tag{2.6}$$

where the chiral anomaly and gravitational anomaly coefficients are:

$$\mathcal{C} = \frac{1}{2\pi^2}, \quad \mathcal{D} = \frac{1}{6}.\tag{2.7}$$

Hydrodynamic excitations are low energy long wavelength excitations around the equilibrium state in fluid. To find them, one has to firstly choose a set of hydrodynamic variables and then linearize the equations of motion in terms of their fluctuations around a thermodynamic solution. It is conventional to consider the microscopic conserved quantities and choose the hydro variables associatively, like

$$\phi_a = (\epsilon, \boldsymbol{\pi}_i, n, n_5), \quad a = 1, 2, \dots, 6\tag{2.8}$$

where ϵ , $\boldsymbol{\pi}$ and n have microscopic definitions given by $T^{00}(x)$, $T^{0i}(x)$ and $J^0(x)$ [19, 20]. However, we prefer to choose ϕ_a as it follows:

$$\phi_a = (T, \boldsymbol{\pi}_i, \mu, \mu_5), \quad a = 1, 2, \dots, 6\tag{2.9}$$

where $\pi_i = wv_i$ and $w = \epsilon + p$ is the enthalpy density. This special choice allows us to compute the hydrodynamic modes for a fluid with general equation of state. However, it has to be noted that in this paper we focus just on the 4-dimensional conformal fluid case with the following equation of state

$$\epsilon = \frac{1}{c_s^2} p, \quad \left(c_s = \frac{1}{\sqrt{3}} \right).\tag{2.10}$$

However in our future computations, it will be needed to express the dynamical fields ϵ , n and n_5 in terms of the variables (2.9). To proceed, we need the following thermodynamic derivatives

$$\begin{aligned}\alpha_1 &= \frac{\partial \epsilon}{\partial T}, & \alpha_2 &= \frac{\partial \epsilon}{\partial \mu}, & \alpha_3 &= \frac{\partial \epsilon}{\partial \mu_5} \\ \beta_1 &= \frac{\partial n}{\partial T}, & \beta_2 &= \frac{\partial n}{\partial \mu}, & \beta_3 &= \frac{\partial n}{\partial \mu_5} \\ \gamma_1 &= \frac{\partial n_5}{\partial T}, & \gamma_2 &= \frac{\partial n_5}{\partial \mu}, & \gamma_3 &= \frac{\partial n_5}{\partial \mu_5}\end{aligned}\tag{2.11}$$

which constitute the susceptibility matrix as

$$\tilde{\chi} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

Let us recall that the elements of this matrix are not generally independent; using the thermodynamics relations one simply show that:

$$\beta_1 = \frac{1}{c_s^2} \frac{n}{T} - \beta_2 \frac{\mu}{T} - \gamma_2 \frac{\mu_5}{T} \quad (2.12)$$

$$\beta_3 = \gamma_2 \quad (2.13)$$

$$\gamma_1 = \frac{1}{c_s^2} \frac{n_5}{T} - \beta_3 \frac{\mu}{T} - \gamma_3 \frac{\mu_5}{T}. \quad (2.14)$$

The equilibrium state of the system is specified with⁴

$$\begin{aligned} u^\mu &= \left(1, \boldsymbol{\Omega} \times \boldsymbol{x}\right) \quad \Omega r \ll 1, \\ T &= \text{Const.}, \quad \mu = \text{Const.}, \quad \mu_5 = \text{Const.} \\ \boldsymbol{B} &= \text{Const.} \end{aligned} \quad (2.15)$$

with the pressure $p = p(T, \mu, \mu_5)$ satisfying:

$$dp = s dT + n d\mu + n_5 d\mu_5. \quad (2.16)$$

In this paper, we compute six hydrodynamic modes associated with six hydro variables in three different cases. We first consider hydro excitations in the equilibrium of the QCD fluid coupled to an external magnetic field ($\boldsymbol{\Omega} = 0$). We then turn off the magnetic field and consider the hydro excitations in the QCD matter rotating with a constant vorticity ($\boldsymbol{B} = 0$). Finally, the hydro modes are studied in rotating QCD fluid coupled to an external magnetic field. To be more relevant to heavy ion collision case, we re-express our results in the limit where chiral charge density vanishes.

2.2 Results Addressed

In what follows in this subsection, we review and address all of our formulas found in the paper. For this section to be self-organised, we also try to give a brief physical interpretation for each case.

2.2.1 General Results

Our results may be divided into three parts which we have listed them in Table 1 below.

⁴We use the metric $g_{\mu\nu} = (-1, 1, 1, 1)$ in this paper.

First, we have considered the non-rotating QCD fluid coupled to an external magnetic field. We show that in this case, two scalar modes (3.6) propagate in the fluid with different velocities. These are actually the generalization of the CMWs firstly found in [28]. We show that the difference between velocities of the fast and slow CMWs, which distinguishes our result from that of [28], is just due to presence of a finite charge density in the fluid, either vector charge or axial one.

| | $\omega_{1,2}$ | $\omega_{3,4}$ | $\omega_{5,6}$ |
|-----------------|--|-----------------------------|----------------|
| $B \neq 0$ | Chiral Magnetic Wave (3.6) | mixed Alfvén-Sound | |
| $\Omega = 0$ | | (3.25) | (3.26) |
| $B = 0$ | Chiral Vortical Wave (4.10) | mixed Coriolis-Sound | |
| $\Omega \neq 0$ | | (4.16) | (4.17) |
| $B \neq 0$ | mixed Chiral Magnetic-Vortical Wave (5.5) | mixed Alfvén-Coriolis-Sound | |
| $\Omega \neq 0$ | | (5.8) | |

Table 1. Hydro modes in rotating QCD fluid coupled to an external magnetic field.

In addition to CMWs, there are another four hydrodynamic modes (3.25), (3.26) in this case. These modes are all mixed scalar-vector waves. At zero order in derivative expansion, each of them is a mixture of ordinary sound with Larmore frequency, reminiscent of the magnetosonic waves in the ideal magnetohydrodynamics [33]. We will explain in detail that in contrast to the latter case, magnetosonic waves become dispersive when they are coupled to a non-dynamical magnetic field.

At first order in derivative expansion, all the above mixed modes get corrections proportional to magnetic field with the coefficient of chiral anomaly. The situation is actually analogous to what appears for the velocity of Chiral Alfvén Waves (CAW) in a chiral fluid of single right-handed fermions [34, 35]. One important difference between the result of [34, 35] and our result in the current paper is that, here, the chiral anomaly coefficients appears in the expression of the velocity, as well as the the gravitational anomaly coefficient \mathcal{D} . This is simply due to this fact that the coefficient of vorticity term in the vector current, namely ξ , depends on the chiral anomaly coefficient \mathcal{C} as well as the gravitational coefficient \mathcal{D} . Since the propagation of the CAW is originated from the coupling of the vorticity term and the magnetic field, it is natural to expect that the velocity of CAW depends on both anomaly coefficients.

In the special limit where the wave vector is parallel to the magnetic field, all the six modes given in first row of Table 1 are distinguishable as it follows: in addition to two CMWs, the four mixed waves split into exactly two ordinary sound waves together with two CAWs.

Second, we consider QCD fluid with constant vorticity in the absence of any external electromagnetic field. Similar to the **first** case discussed above, we find two scalar excitations together with four mixed scalar-vector modes, around equilibrium state of the fluid. The scalar modes are the generalizations of the CVWs found in [32] to the case of a fluid with non-zero axial charge density. It will be shown that as long as the axial charge density is non-vanishing, one of the CVWs moves faster than the other one. It should be also noted that the velocity of CVWs depends, in general, on both the chiral anomaly and the gravitational anomaly coefficients. The scalar-vector modes, however, carry no information about the quantum field theory anomalies. These are actually mixed modes of ordinary sound together with the inertial (or Coriolis) waves. Since the fluid is not coupled to any magnetic field, no Lorentz force acts on the system and therefore no anomaly effect appears in these four modes. As a result, compared to the **first** case, CAW can not propagate in the **second** case.

The results concerned with the **third** case have been addressed in the last row of Table 1. They correspond to a more general case wherein both vorticity and external magnetic field are present. In the pure scalar sector, CMW and CVW are mixed with each other into two new dispersive modes. As one expects, the remaining four modes are the mixed Sound-Alfvén-Coriolis modes. Let us denote that the dispersion relations of modes addressed in this case are in general so complicated, so we do not write their velocities explicitly anywhere in this paper. In some special cases like the QCD fluid in heavy ion collisions, however, these expressions become shorter and more understandable. Relatedly, before ending the current section we come back to this point and give the corresponding results associated with the quark gluon plasma, in an independent subsection.

2.2.2 Special Limits

In this subsection, we give the velocity of hydro modes associated with three cases discussed above for special directions of propagation. In Table 2 we have listed the hydrodynamic modes with their velocity of propagation in QCD hot fluid coupled either parallel or transverse to an external magnetic field.

Analogous to the case of a chiral fluid with single chirality [35], the anomaly effects can not be detected in directions transverse to the magnetic field here. The only modes propagating in transverse directions are the magnetosonic waves. By magnetosonic wave here, however, we do not mean precisely the familiar magnetosonic waves in the ideal magnetohydrodynamics. As it is well-known in magnetohydrodynamics, the pressure perturbations produced by Maxwell dynamics intensify the fluid pressure perturbations, resulting in an excess in the velocity of sound. The latter is a purely magnetohydrodynamical effect by this mean that the magnetic pressure perturbations are produced through compression and rarefaction of the magnetic

| | $\mathbf{B} \parallel \mathbf{k}$ | $\mathbf{B} \perp \mathbf{k}$ |
|----------------------|--|---|
| Chiral Magnetic Wave | $v_{1,2} = -\frac{\mathcal{A}_1 \pm \sqrt{\mathcal{A}_1^2 - \alpha_1 \mathcal{C}^2 \mathcal{E}}}{\mathcal{E}} B$ | — |
| Magnetosonic Wave | $v_{3,4} = \pm c_s$ | $v_{3,4} = \pm \frac{c_s^2 k}{\sqrt{c_s^2 k^2 + \Omega_L^2}}$ |
| Chiral Alfvén Wave | $v_{5,6} = -\frac{\xi}{w} B$ | — |

Table 2. The velocity of hydrodynamic modes in QCD fluid coupled to an external magnetic field.

field lines. In our case, however, the effect of the magnetic field is just to exert the opposite Lorentz forces on a momentum perturbation and eventually decrease the hydrodynamical pressure. We will argue in detail in the text why the magnetosonic waves here, are not only dispersive but also have velocities increasing with the wave number.

The situation is different in the direction parallel to the magnetic field. In this case, all six possible long wave length modes may be excited; right- and left-moving sounds as well as two degenerate CAWs and two CMWs. In contrast to a single right-handed chiral fluid, CAWs in QCD fluid do not propagate at zero vector density limit. Regarding the CMWs, it should be mentioned that there may exist two of them in a QCD type fluid: fast and slow CMWs. In particular, when $n_5 \rightarrow 0$, the anomaly expression \mathcal{A}_1 vanishes and the fast and slow modes become left-right moving modes with the same velocities.

Let us now focus on the velocities of the hydro modes propagating parallel or transverse to the vorticity in a rotating fluid (see Table 3). As the first point, one can see that the anomaly effects can not be detected in the directions transverse to the vorticity. In addition, the Coriolis force acts in the opposite direction of the propagation and consequently, decreases the pressure; so the sound modes move slower here than in a non-rotating fluid. Sound waves also become dispersive due to rotation of the fluid.

Parallel to the vorticity, in addition to ordinary sound modes, two other modes may be excited. These modes are both, CVWs which are actually the coherent perturbations of the vector, axial and energy currents at finite axial and vector charge densities. While these two CVWs propagate with different velocities in general, in the non-chiral limit, $n_5 = 0$, their velocities will be the same. Let us denote that our result in the non-chiral limit differs from that of found in [32]. The reason is that the authors of [32] have considered a QCD type fluid with fixed temperature in which, no fluctuations of temperature can contribute to the propagation of CVWs ⁵. We

⁵Let us remind that as it can be seen in Table 3, in agreement with our earlier explanation, CAWs could not propagate in QCD rotating fluid.

| | $\Omega \parallel \mathbf{k}$ | $\Omega \perp \mathbf{k}$ |
|----------------------|--|---|
| Chiral Vortical Wave | $v_{1,2} = -\frac{\mathcal{A}_3 \pm \sqrt{\mathcal{A}_3^2 - 4\mathcal{E}\mathcal{A}_4}}{\mathcal{E}} \Omega$ | — |
| Coriolis-Sound | $v_{3,4} = \pm c_s$ | $v_{3,4} = \pm \frac{c_s^2 k}{\sqrt{c_s^2 k^2 + \Omega^2}}$ |

Table 3. The velocity of hydrodynamic modes in QCD rotating fluid with vorticity Ω .

show that our formalism is simply able to reproduce the result of [32] by keeping the temperature fixed.

2.2.3 Results Associated with Quark Gluon Plasma

One important part of our results in this paper is related to QCD fluid which is not only rotating but is also coupled to an external magnetic field. As it was mentioned earlier, the dispersion relations of hydro modes in this general situation are complicated. To be more applicable and comparable with experimental observations, we have applied our general formulas (addressed in the last row of Table 1) to the special case of heavy ion collisions and rewritten the results for this special case as well.

As it is well-known, due to non-central collisions of nuclei in heavy ion experiments, a magnetic field is produced parallel to the vorticity of hot plasma. It is actually an example of the general situation mentioned in previous paragraph. Phenomenologically, it would be interesting to investigate how the hydrodynamic waves propagating on top of this flowing plasma may affect the physical observables, e.g. the final spectrum of the charged particles. However since the fluid is not in global equilibrium, the analytic study of the modes is challenging (see [36]). Instead, we specifically compute the hydro modes in the equilibrium state for quark gluon matter and use them to obtain new phenomenological information about heavy ion physics.

It is well-accepted that the plasma of quarks and gluons produced in heavy ion collision experiments is initially non-chiral, i.e. $n_5 = 0$. So according to our discussion below Table 1, chiral Alfvén wave does not propagate in heavy ion matter while, CMW, CVW and sound waves can all propagate around equilibrium state. Our novel result in this case is related to the mixed CMVWs. As we mentioned in previous subsection, CMW is the mixture between CMW and CVW. We show that at finite vector and axial chemical potentials, CMVWs propagate with following velocities:

$$v_{CMVW} = \pm \left(v_{CVW} + v_{CMW} \right) \quad (2.17)$$

where v_{CVW} and v_{CMW} are

$$v_{CMW} = \frac{\mathcal{C} B}{\chi} \frac{\left(1 - \frac{n\mu}{w}\right)}{\mathcal{K}}, \quad (2.18)$$

$$v_{CVW} = \frac{\mathcal{C} \mu \Omega}{\chi} \frac{\left(1 - \frac{n\mu}{w} - \frac{\mathcal{D}}{\mathcal{C}} \frac{nT^2}{\mu w}\right)}{\mathcal{K}} \quad (2.19)$$

and

$$\mathcal{K} = \left(1 - \frac{n\mu}{w} - \frac{nT}{w\chi} \frac{\partial n}{\partial T}\right)^{1/2}. \quad (2.20)$$

Although the equality (2.17) is well-known in a fluid with only axial and vector currents [37], we have shown that it would be also valid when the energy and momentum perturbations are taken into account as well. It should be noted that v_{CMW} was firstly computed at $\mu = \mu_5$ in [28] and the associated expression given in (2.18) is the generalization of the results of [28] to the case of finite vector charge density with considering the perturbations of energy and momentum in the fluid.

In the case of v_{CVW} , the previous computations in the literature were limited to the case with the energy and momentum perturbations in the fluid forced to be turned off [32], although a more general case with considering also the energy perturbations have been studied in [37]. Our computations however include the perturbations of energy and momentum as well and the second multiplying fraction in (2.19) is the new contribution originated from these perturbations in the fluid.

There is an interesting point about equation (2.17) above. Our computations show that the velocity of CVW at finite density depend on the coefficient of gravitational anomaly as well as the coefficient of the chiral anomaly. As a result, we find that v_{CMW} which was thought that does not depend on gravitational anomaly coefficient so far, is actually a function of both \mathcal{D} and \mathcal{C} . This means that by studying the effect of the chiral waves on the final spectrum of the charged particle in QGP, it might be possible to investigate the presence of gravitational anomaly.

3 QCD Fluid Coupled to External Magnetic Field

In this section we consider a QCD type fluid coupled to an external magnetic field and compute the spectrum of its hydrodynamic excitations in detail. After deriving the covariant linearized equations, we divide our computations into two parts. First, we consider pure scalar perturbations and then in another subsection, we take the mixed scalar-vector perturbations under study. To be more complete and clear, we also discuss on the Riemann invariants of the fluid and show, to each hydro excitation, which coherent combination of perturbations may correspond.

3.1 Equations of Motion Linearized

Let us consider the hydro field defined in (2.9) is slightly deviated from its thermodynamic value as the following:

$$\phi_a + \delta\phi_a = \left(T + \delta T, \mathbf{0} + \boldsymbol{\pi}, \mu + \delta\mu, \mu_5 + \delta\mu_5 \right). \quad (3.1)$$

To first order in δ variations, the equations of motion may be covariantly written as

$$M_{ab}^B(\mathbf{k}, \omega) \delta\phi_a(\mathbf{k}, \omega) = 0 \quad (3.2)$$

with M_{ab}^B given by:

$$\begin{bmatrix} -i\alpha_1\omega & ik_j & -i\alpha_2\omega & -i\alpha_3\omega \\ i\alpha_1 v_s^2 k^i & -i\omega\delta_j^i - i\frac{\xi_s}{2\bar{w}}(\mathbf{B} \cdot \mathbf{k}\delta_j^i - B_j k^i) - \frac{\bar{n}}{\bar{w}}\epsilon^i{}_{jl}B^l & i\alpha_2 v_s^2 k^i & i\alpha_3 v_s^2 k^i \\ -i\beta_1\omega + \left(\frac{\partial\xi_B}{\partial T}\right)i\mathbf{B} \cdot \mathbf{k} & \frac{\bar{n}}{\bar{w}}ik_j - \frac{\xi_B}{\bar{w}}i\omega B_j & -i\beta_2\omega + \left(\frac{\partial\xi_B}{\partial\mu}\right)i\mathbf{B} \cdot \mathbf{k} & -i\beta_3\omega + \left(\frac{\partial\xi_B}{\partial\mu_5}\right)i\mathbf{B} \cdot \mathbf{k} \\ -i\gamma_1\omega + \left(\frac{\partial\xi_{5B}}{\partial T}\right)i\mathbf{B} \cdot \mathbf{k} & \frac{\bar{n}_5}{\bar{w}}ik_j - \frac{\xi_{5B}}{\bar{w}}i\omega B_j & -i\gamma_2\omega + \left(\frac{\partial\xi_{5B}}{\partial\mu}\right)i\mathbf{B} \cdot \mathbf{k} & -i\gamma_3\omega + \left(\frac{\partial\xi_{5B}}{\partial\mu_5}\right)i\mathbf{B} \cdot \mathbf{k} \end{bmatrix}.$$

As it can be obviously seen above, none of the elements of matrix M_{ab} vanishes in general. It means that each of the hydrodynamic excitations in this system might be a coherent excitation of all scalar and vector perturbations. Via computing the Riemann invariants, however, one notices that there are two kinds of modes propagating in the fluid: two waves of the scalar perturbations and four waves of the scalar-vector perturbations⁶.

3.2 Characteristics and Riemann Invariants

Before starting to compute the hydrodynamic modes, let us briefly review the notion of characteristics and the Riemann invariants in fluid dynamics. As it is well-known, in a fluid whose space of states is D -dimensional (in our case $D = 6$), there exist in general D characteristics or equivalently D hydrodynamic waves. These characteristics describe the different ways through which, a small perturbation in the state of fluid may propagate in the space-state. To each of characteristics, one family of integral curves in the state-space is corresponded. Those perturbations that propagate only through curves of one characteristic family correspond to the Riemann invariants [13]. So the Riemann invariant \mathcal{R}_i associated with the hydro mode ω_i satisfies the following equation:

$$\left(\partial_t + \mathbf{v}_i \cdot \boldsymbol{\nabla} \right) \mathcal{R}_i = 0 \quad (3.3)$$

where \mathbf{v}_i is the velocity of i^{th} mode ω_i .

⁶Throughout this paper wherever we mention scalar or vector, we mean the representations of $SO(3)$ spatial rotational group.

In order to determine the Riemann invariants, one assumes that the linear equations of perturbations may be written as the following:

$$\partial_t \delta\phi_a(\mathbf{k}, t) + D_{ab}(\mathbf{k}) \delta\phi_b(\mathbf{k}, t) = 0. \quad (3.4)$$

Firstly, it is needed to compute the eigenvalues of the matrix D as the characteristics or equivalently the hydrodynamic modes. The next step is to compute the eigenvectors of matrix D . Then the Riemann invariant associated with each of these vectors is the special scalar combination of the components of $\delta\phi_a$ which remains invariant along the integral curve generated by that eigenvector in the space-state. It should be denoted that in Fourier space, $D_{ab} = M_{ab} + i\omega\delta_{ab}$ where M_{ab} was defined in (3.2).

Let us mention that computing the Riemann invariant for our fluid with a 6-dimensional state-space is so complicated. For this reason, in the following, we restrict ourself to compute the eigenvectors corresponding to the characteristics of the fluid. Having the eigenvectors, one might be able to figure out which perturbations are carried with any of the hydrodynamic modes.

3.3 Scalar Sector: Chiral Magnetic Waves

Among the eigen vectors $\delta\tilde{\phi}_i = (\delta T, \boldsymbol{\pi}, \delta\mu, \delta\mu_5)$ of matrix D , the following pair lies in the 3-dimensional scalar subspace of state-space, spanned by δT , δn and δn_5

$$\delta\tilde{\phi}_{1,2} = \left(r \frac{\alpha_2}{\alpha_1} + s \frac{\alpha_3}{\alpha_1}, \mathbf{0}, -r, -s \right), \quad (3.5)$$

with r and s arbitrary parameters. These are actually coherent perturbations of energy, vector and axial charge currents, corresponding to the following characteristics

$$\omega_{1,2}(k) = -\frac{\mathcal{A}_1 \pm \sqrt{\mathcal{A}_1^2 - \mathcal{A}_2 \mathcal{E}}}{\mathcal{E}} \mathbf{B} \cdot \mathbf{k}. \quad (3.6)$$

In the above formula, we have defined

$$\mathcal{E} = -\epsilon^{ijk} \alpha_i \beta_j \gamma_k \quad (\epsilon^{123} = 1) \quad (3.7)$$

with anomaly expressions ⁷

$$\mathcal{A}_0 = +\frac{n\mu}{w} \alpha_{[1}\gamma_{2]} + \frac{n_5\mu_5}{w} \alpha_{[1}\beta_{3]} - \frac{n\mu_5}{w} \alpha_{[1}\gamma_{3]} - \frac{n_5\mu}{w} \alpha_{[1}\beta_{2]} \quad (3.8)$$

$$\mathcal{A}_1 = \frac{1}{4\pi^2} \left(\alpha_{[3}\beta_{1]} + \alpha_{[2}\gamma_{1]} + \frac{2\mu\mu_5}{w} \mathcal{E} + \mathcal{A}_0 \right) \quad (3.9)$$

$$\mathcal{A}_2 = \frac{\alpha_1}{4\pi^4} \left(1 - \frac{n\mu + n_5\mu_5}{w} \right) + \frac{\mu\mu_5}{4\pi^4 w} \left(\alpha_{[3}\beta_{1]} + \alpha_{[2}\gamma_{1]} + \frac{\mu\mu_5}{w} \mathcal{E} + \mathcal{A}_0 \right) \quad (3.10)$$

⁷ $A_{[i}B_{j]} = A_i B_j - A_j B_i$.

Let us recall that by using the thermodynamic transformations

$$\delta T = \left(\frac{\partial T}{\partial \epsilon} \right) \delta \epsilon + \left(\frac{\partial T}{\partial n} \right) \delta n + \left(\frac{\partial T}{\partial n_5} \right) \delta n_5 \quad (3.11)$$

$$\delta \mu = \left(\frac{\partial \mu}{\partial \epsilon} \right) \delta \epsilon + \left(\frac{\partial \mu}{\partial n} \right) \delta n + \left(\frac{\partial \mu}{\partial n_5} \right) \delta n_5 \quad (3.12)$$

$$\delta \mu_5 = \left(\frac{\partial \mu_5}{\partial \epsilon} \right) \delta \epsilon + \left(\frac{\partial \mu_5}{\partial n} \right) \delta n + \left(\frac{\partial \mu_5}{\partial n_5} \right) \delta n_5, \quad (3.13)$$

one can alternatively express the eigenvectors (3.5) in terms of another set of fluctuations $\delta \epsilon$, δn and δn_5 . These two modes are in fact the generalizations of left- and right-moving CMWs [28] to the case of a fluid with finite vector and axial charge densities.

The transport of the energy (or temperature) perturbations by the CMWs (3.6) is actually a new feature which may distinguish them from those originally found in [28]. To make the difference clear, let us go back and consider the eigenvectors (3.5). The thermodynamic coefficients $\alpha_2 = \partial \epsilon / \partial \mu$ and $\alpha_3 = \partial \epsilon / \partial \mu_5$ are non-vanishing at finite vector and axial charge densities, so one would expect only at $n = n_5 = 0$ [28], the temperature perturbation is not carried by the CMWs (see equation (3.5)). Additionally, while both the left- and right-moving CMWs in [28] are identified with one velocity, the velocity of our two CMWs is not the same; it is in fact the manifestation of the energy transport by the CMWs.

In order to compare the fast and slow CMWs, we compute their velocities of propagation in the special case of a fluid at small vector and axial density. In such situation, the thermodynamic pressure of a conformal fluid may be expanded in a power series of chemical potentials of right- and left-handed parts [36]:

$$P(T, \mu_L, \mu_R) = c_0 T^4 + c_1 T^2 (\mu_L^2 + \mu_R^2) + c_2 T^4 (\mu_L^4 + \mu_R^4) + \dots \quad (3.14)$$

where by use of $\mu = \frac{1}{2} (\mu_L + \mu_R)$ and $\mu_5 = \frac{1}{2} (\mu_L - \mu_R)$, one may rewrite the pressure as

$$P(T, \mu, \mu_5) = c_0 T^4 + 2c_1 T^2 (\mu^2 + \mu_5^2) + 2c_2 (\mu^4 + 6\mu^2 \mu_5^2 + \mu_5^4) + \dots \quad (3.15)$$

Using this, the components of matrix (2.11) may be written as

$$\begin{aligned} \alpha_1 &= 12T(c_0 T^2 + c_1(\mu^2 + \mu_5^2)), & \alpha_2 &= 12\mu(c_1 T^2 + 2c_2(\mu^2 + 3\mu_5^2)), & \alpha_3 &= \frac{\mu_5}{\mu} \alpha_2 \\ \beta_1 &= 8c_1 T \mu, & \beta_2 &= 4(c_1 T^2 + 6c_2(\mu^2 + \mu_5^2)), & \beta_3 &= 48c_2 \mu \mu_5, \\ \gamma_1 &= 8c_1 T \mu_5, & \gamma_2 &= \beta_3, & \gamma_3 &= \beta_2. \end{aligned} \quad (3.16)$$

In this limit we obtain:

$$\begin{aligned} \mathcal{A}_1 &= -\frac{288c_0 c_2}{\pi^2} \mu \mu_5 T^3 \\ \mathcal{A}_2 &= \frac{3c_0}{\pi^2} T^3 \end{aligned} \quad (3.17)$$

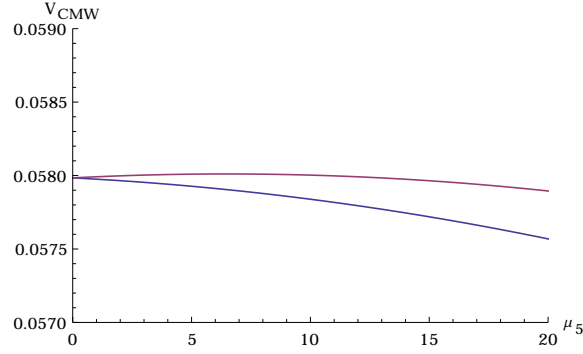


Figure 1. The velocity of fast and slow CMWs versus the axial chemical potential. Notice that in the situation we are considering, these two waves propagate in opposite directions.

As a result, the velocity of fast and slow CMWs take the following form:

$$\begin{aligned} v_{slow} &= \frac{B}{8\pi c_1 T^2} \left(1 + \left(\frac{c_1}{2c_0} - \frac{6c_2}{c_1} \right) \frac{\mu^2 + \mu_5^2}{T^2} - \frac{12c_2}{\pi c_1} \frac{\mu\mu_5}{T^2} \right) \\ |v_{fast}| &= \frac{B}{8\pi c_1 T^2} \left(1 + \left(\frac{c_1}{2c_0} - \frac{6c_2}{c_1} \right) \frac{\mu^2 + \mu_5^2}{T^2} + \frac{12c_2}{\pi c_1} \frac{\mu\mu_5}{T^2} \right). \end{aligned} \quad (3.18)$$

In order to numerically evaluate the velocity of the fast and slow CMWs, we focus on the heavy ion fluid specifically. As it is well-understood, the QCD fluid in heavy ion experiments has a small vector charge density and is also initially non-chiral. However, via the chiral magnetic effect, the axial charge density may fluctuate and get non-zero value when the plasma is flowing. This may allow us to approximate the system with a fluid at small vector and axial density. Using the conventional values for physical quantities $T = 170$ MeV, $\sqrt{B} = 130$ MeV and $\frac{\mu}{T} = 0.1$, with c_0 and c_1 given in [36]

$$\begin{aligned} c_0 &= 0.8 \left(\frac{8\pi^2}{45} + \frac{\pi^2}{15} \left(\frac{7}{4} \right) n_f \right) \\ c_1 &= 0.8 \left(\frac{n_f}{4} \right) \\ c_2 &= 1.0 \left(\frac{1}{4\pi^2} \right) \end{aligned} \quad (3.19)$$

we have plotted the velocity of CMWs versus the axial chemical potential μ_5 , for $n_f = 2$ in figure 1. As it is clear from the plot, for μ_5 less than 10 MeV⁸, i.e. the range in which the perturbation theory might better work, the velocity of the fast CMW increases with the axial density while the velocity of the slow one is decreasing. Interestingly at $\mu_5 = 0$, \mathcal{A}_1 vanishes and consequently both velocities coincide with

⁸We get the upper bound for the perturbation theory being $\mu/T \sim \mu_5/T \sim 0.1$.

each other (see (3.18))

$$v = \pm \frac{B}{8\pi c_1 T^2} \left(1 + \left(\frac{c_1}{2c_0} - \frac{6c_2}{c_1} \right) \frac{\mu^2}{T^2} \right) = v_{KY} \left(1 + \left(\frac{c_1}{2c_0} - \frac{6c_2}{c_1} \right) \frac{\mu^2}{T^2} \right) \quad (3.20)$$

with v_{KY} the velocity of CMW found in [28] in case of QCD fluid at $n = n_5 = 0$.

In summary, at small vector and axial chemical potentials (compared to T), one of the CMWs propagates faster than the other. Particularly, when $n_5 = 0$ the velocities of CMWs become degenerate and less than the degenerate velocity of CMWs when the vector charge density vanishes as well⁹. The latter point is something exclusive for a QCD type fluid with the pressure given by (3.14). In a fluid with a different equation of state, the result may differ from (3.20) [10].

3.4 Mixed Scalar-Vector Sector

In addition to two scalar modes computed in previous subsection, there are yet four modes which has to be identified. Computing the eigen-vectors of the matrix M_{ab}^B , we find for $i = 3, 4, 5, 6$:

$$\begin{aligned} \delta\tilde{\phi}_i &= \left(\delta T, \delta\pi, \delta\mu, \delta\mu_5 \right) \\ &= \left(1, -\frac{w\mathcal{E}\omega_i^{(1)}\mathbf{k}}{C_2\mathbf{k}^2} + \frac{n\mathcal{E}\left(i(\omega_i^{(1)})^2(\mathbf{B}\times\mathbf{k}) + \frac{n}{w}\omega_i^{(1)}(\mathbf{B}\cdot\mathbf{k})\mathbf{B} - \frac{n}{w}\omega_i^{(1)}(\mathbf{B}\cdot\hat{\mathbf{k}})^2\mathbf{k} \right)}{C_2((\omega_i^{(1)})^2\mathbf{k}^2 - n^2/w^2(\mathbf{B}\cdot\mathbf{k})^2)}, \frac{C_1}{C_2}, \frac{C_3}{C_2} \right) \end{aligned} \quad (3.21)$$

with

$$C_1 = n\alpha_{[1}\gamma_3] - n_5\alpha_{[1}\beta_3] - w\beta_{[1}\gamma_3] \quad (3.22)$$

$$C_2 = n\alpha_{[3}\gamma_2] - n_5\alpha_{[3}\beta_2] - w\beta_{[3}\gamma_2] \quad (3.23)$$

$$C_3 = n\alpha_{[2}\gamma_1] - n_5\alpha_{[2}\beta_1] - w\beta_{[2}\gamma_1] \quad (3.24)$$

where in the expression (3.21), $\omega_i^{(1)}$ and $\omega_i^{(2)}$ are the first and second order derivative parts of the dispersion relations of ω_i ¹⁰ given below. Before proceeding further, let us denote that the above eigenvectors have generally non-vanishing scalar and vector components in the state-space. For this reason, we refer to the current sector as the scalar-vector sector.

⁹ $\frac{9}{2c_0} \frac{c_1}{c_1} - \frac{6c_2}{c_1} < 0$.

¹⁰ In the expressions (3.25) and (3.26), $\omega_i^{(1)}$ and $\omega_i^{(2)}$ are in fact computed from the zero and first order derivative parts of constitutive relations respectively.

The associated dispersion relations are as the following

$$\begin{aligned}
\omega_{3,4} &= \omega_{3,4}^{(1)} + \omega_{3,4}^{(2)} \\
&= \pm \frac{1}{\sqrt{2}} \sqrt{c_s^2 k^2 + \Omega_L^2 + \sqrt{(c_s^2 k^2 + \Omega_L^2)^2 - 4c_s^2 k^2 \Omega_L^2 \cos^2 \theta}} \\
&\quad - \frac{\Omega_L^2 \left(c_s^2 k^2 + \Omega_L^2 + \sqrt{(c_s^2 k^2 + \Omega_L^2)^2 - 4c_s^2 k^2 \Omega_L^2 \cos^2 \theta} - 2c_s^2 k^2 \cos^2 \theta \right) \cos \theta}{(c_s^2 k^2 + \Omega_L^2) \left(c_s^2 k^2 + \Omega_L^2 + \sqrt{(c_s^2 k^2 + \Omega_L^2)^2 - 4c_s^2 k^2 \Omega_L^2 \cos^2 \theta} \right) - 4c_s^2 k^2 \Omega_L^2 \cos^2 \theta} (\xi B) k
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\omega_{5,6} &= \omega_{5,6}^{(1)} + \omega_{5,6}^{(2)} \\
&= \pm \frac{1}{\sqrt{2}} \sqrt{c_s^2 k^2 + \Omega_L^2 - \sqrt{(c_s^2 k^2 + \Omega_L^2)^2 - 4c_s^2 k^2 \Omega_L^2 \cos^2 \theta}} \\
&\quad - \frac{\Omega_L^2 \left(c_s^2 k^2 + \Omega_L^2 - \sqrt{(c_s^2 k^2 + \Omega_L^2)^2 - 4c_s^2 k^2 \Omega_L^2 \cos^2 \theta} - 2c_s^2 k^2 \cos^2 \theta \right) \cos \theta}{(c_s^2 k^2 + \Omega_L^2) \left(c_s^2 k^2 + \Omega_L^2 - \sqrt{(c_s^2 k^2 + \Omega_L^2)^2 - 4c_s^2 k^2 \Omega_L^2 \cos^2 \theta} \right) - 4c_s^2 k^2 \Omega_L^2 \cos^2 \theta} (\xi B) k
\end{aligned} \tag{3.26}$$

In the equations given above, Ω_L is the Larmore frequency as being

$$\Omega_L = \frac{nB}{w}. \tag{3.27}$$

Before studying the modes given by (3.25) and (3.26) in general, we first analyse them at the zero axial charge density limit. When $\mu_5 = 0$, the second terms in (3.25) and (3.26) vanish and just the square root terms contribute. To clarify the difference between $\omega_{3,4}$ and $\omega_{5,6}$ modes in this limit, let us now consider two special cases in the following:

• **$B \parallel \mathbf{k}$**

Consider $\mathbf{k} = (0, 0, k)$, in this case the dispersion relations and the corresponding eigenvectors of the four modes above may be written as it follows for $c_s k > \Omega_L$:

$$\begin{aligned}
\omega_{3,4} &= \pm c_s k, & \delta \tilde{\phi}_a &= \left(1, 0, 0, \mp \frac{w\mathcal{E}}{C_2} c_s, \frac{C_1}{C_2}, \frac{C_3}{C_2} \right) \\
\omega_{5,6} &= \pm \Omega_L, & \delta \tilde{\phi}_a &= (0, 1, \pm i, 0, 0, 0)
\end{aligned} \tag{3.28}$$

It is obvious that the modes $\omega_{3,4}$ are longitudinal left- and right-moving sound modes, while the modes $\omega_{5,6}$ are two oppositely polarized non-propagating vortices. As mentioned above, $\omega_{5,6}$ are the frequencies related to Larmore rotation of a charged fluid in the magnetic field.

• $\mathbf{B} \perp \mathbf{k}$

In this case the modes $\omega_{5,6}$ are not excited, however for the other two modes we have:

$$\omega_{3,4} = \pm \sqrt{c_s^2 k^2 + \Omega_L^2}, \quad \delta\tilde{\phi}_a = \left(1, \frac{w\mathcal{E}}{C_2 k^2} \left(\frac{in}{w} \mathbf{B} \times \mathbf{k} \mp \sqrt{c_s^2 k^2 + \Omega_L^2} \mathbf{k} \right), \frac{C_1}{C_2}, \frac{C_3}{C_2} \right) \quad (3.29)$$

These modes, which are mixed of the longitudinal and transverse propagations, are indeed the magnetosonic waves in a thermal plasma and their velocities velocity of these modes

$$v_{3,4} = \pm \frac{c_s^2 k}{\sqrt{c_s^2 k^2 + \Omega_L^2}} \quad (3.30)$$

are always less than the ordinary sound velocity. The reason for this may be explained as it follows. Consider a magnetosonic wave propagating in a special frame wherein $\mathbf{k} = (0, 0, k)$ and $\mathbf{B} = (0, B, 0)$. Now suppose the momentum fluctuation in the $+z$ direction as being $\delta\pi_z$. Thorough acting the Lorentz force on the charged element of the fluid with momentum $\delta\pi_z$, a momentum fluctuation $\delta\pi_x$ is produced in the $-x$ direction. The latter, causes a new Lorentz force acts on the element in the $-z$ direction. This results in decreasing the hydrodynamic pressure pushing the element in the $+z$ direction. As a result, the propagation sound becomes slower. In summary, the velocity fo the magnetosonic waves in a thermal plasma, would be smaller than the velocity of ordinary sound.

Another feature of our magnetosonic waves found above is that, they are dispersive waves with the velocities increasing with the wave number. This means that the constant magnetic field is much more efficient on the propagation of the magnetosonic waves with smaller wave numbers. We actually expected this feature because a constant magnetic field may be regarded as a slowly varying field at infinitely large distances (or small wave numbers) which can not influence the physics in small scales (or large wave number).

Let us now go back and consider the modes (3.25) and (3.26) in a more general situation where the magnetic field is neither parallel nor transverse to the wave vector. A more careful look at the eigen vectors (3.21) shows that not only perturbations of momentum in the directions of both \mathbf{k} and $\mathbf{B} \times \mathbf{k}$ may be carried by our four modes, but the momentum perturbations parallel to the magnetic field may also propagate with these waves. In another word, the projection of the wave vector on the magnetic field is non-vanishing and consequently, the modes have propagating components in the direction of the magnetic field. This is reminiscent of the Alfvén waves in magnetohydrodynamics ¹¹. As a result the four modes $\omega_{3,4}$ and $\omega_{5,6}$ are

¹¹It is worth mentioning that the traditional Alfvén waves are caused by the dynamical magnetic field and may purely propagate parallel to the background magnetic field. In contrary, Alfvén waves in a non-dynamical magnetic field are only detected as mixed with the sound modes in directions neither parallel nor transverse to the magnetic field.

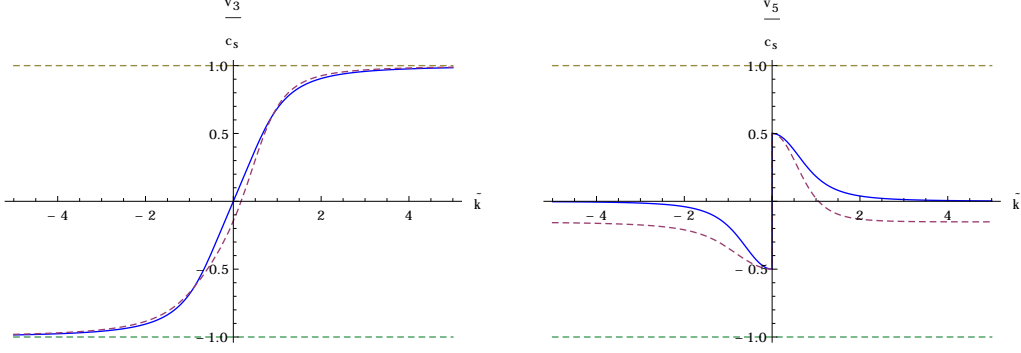


Figure 2. The velocity of mixed modes ω_i : $i = 3, 5$ (normalized by c_s) as the function of dimensionless wave number $\tilde{k} = c_s k / \Omega_L$ for $\cos \theta = 1/2$ and $v_{CAW}^{\parallel} = 0.3$. Blue curves demonstrate the velocities in the non-chiral limit while the dashed curves are related to $\mu_5 \neq 0$.

the mixed Sound-Alfvén waves in general. In figure 2 for the special case where $\hat{\mathbf{B}} \cdot \hat{\mathbf{k}} = \cos \theta = \frac{1}{2}$, we have demonstrated the velocities of the modes ω_3 and ω_5 at $\mu_5 = 0$ as well as in the chiral limit, i.e. $\mu_5 \neq 0$.

The chiral limit results, namely dashed curves in figure 2, needs to be explained in detail. As one may expect, for the transverse case $\mathbf{B} \perp \mathbf{k}$, no new result is obtained; the anomaly is not detected in transverse directions and as before, the magnetosonic waves propagate with the velocity (3.30). In the $\mathbf{B} \parallel \mathbf{k}$ case however, the anomaly effects change the modes as the following for $k c_s > \Omega_L$ ¹²:

$$\begin{aligned} \omega_{3,4} &= \pm c_s k \\ \omega_{5,6} &= \pm \frac{nB}{w} - \frac{\xi}{w} B k \end{aligned} \quad (3.31)$$

with ξ given in (2.6). The modes $\omega_{5,6}$ are propagating waves in the direction of magnetic field; something similar to the traditional Alfvén waves, by this difference that the propagation of these waves here is a pure chiral effect, without need the magnetic field to be dynamical. Moreover, these Chiral Alfvén Waves have also an important difference from those found in [34, 35] for the case of a chiral fluid with single chirality. In the latter case, the CAWs may propagate even at zero axial chemical potential, while in the current work, the fluid has to be at finite axial and vector charge densities to support the propagation of CAWs. Let us denote that the velocity of CAW here

$$\begin{aligned} v_{CAW}^{\parallel} &= - \frac{\xi}{w} B \\ &= - \left\{ \mathcal{C} \left(\mu \mu_5 - \frac{n \mu_5}{3w} (3\mu^2 + \mu_5^2) \right) - \mathcal{D} \frac{n \mu_5}{w} T^2 \right\} \frac{B}{w} \end{aligned} \quad (3.32)$$

¹²In the special case of $\mathbf{B} \parallel \mathbf{k}$, the pair of $\omega_{3,4}$ modes will exchange their characters with those of $\omega_{5,6}$ when the wave number exceeds Ω_L / c_s .

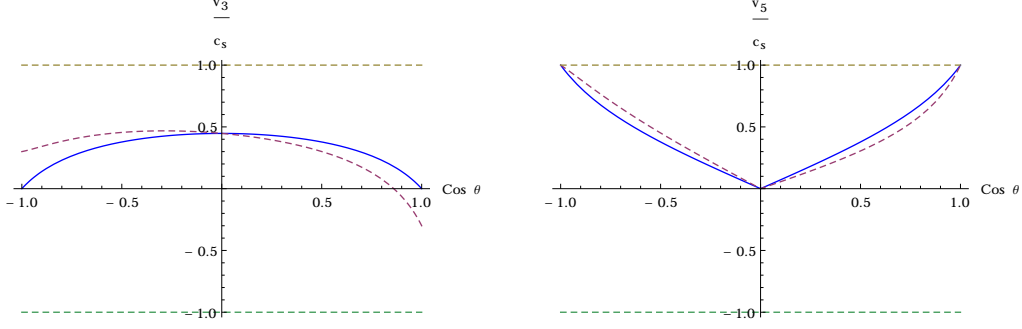


Figure 3. The velocity of mixed modes ω_i : $i = 3, 5$ as the function of $\hat{\mathbf{B}} \cdot \hat{\mathbf{k}} = \cos \theta$. The blue curves are related to the non-chiral limit $\mu_5 = 0$, while the dashed curves demonstrate the velocities at chiral limit for $v_{CAW}^{\parallel} = 0.3$. The horizontal lines show the velocity of these two modes at zero vector density wherein, they are nothing but the ordinary sounds.

depends on both the chiral anomaly and the gravitational anomaly coefficients while, in a fluid with single chirality it was proportional to the gravitational anomaly coefficient [34, 35].

To complete the comparison between the velocity of mixed Sound-Alfvén waves at finite axial chemical potential with their velocity in the non-chiral limit, we have demonstrated the velocities of modes ω_3 and ω_5 (normalized by c_s) versus $\cos \theta = \hat{\mathbf{B}} \cdot \hat{\mathbf{k}}$ in figure 3, for the special wave number $k = \Omega_L/2c_s$.

In both panels, we have plotted the changes in velocity of the waves at zero axial chemical potential (blue curves) together with dashed curves related to the chiral limit. As mentioned earlier, at $\mu_5 = 0$, only the square root terms in (3.25) and (3.26) contribute and consequently, the blue curves are symmetric under $\theta \rightarrow \pi + \theta$ ¹³.

At $\mu_5 \neq 0$, the effect of anomalous transport is entered and the above-mentioned symmetry is broken. Indeed, the asymmetry in the shape of the dashed curves with respect to the transverse axis is the consequence of coupling between the magnetic field and the chiral currents. Precisely speaking, the second terms of (3.25) and (3.26) correspond to the anomaly induced waves propagating opposite to the magnetic field, so for a given $\hat{\mathbf{B}} \cdot \hat{\mathbf{k}} > 0$, the velocity of each mode at $\mu_5 \neq 0$ is less than its velocity at $\mu_5 = 0$ (the situation for $\hat{\mathbf{B}} \cdot \hat{\mathbf{k}} < 0$ is converse). We see that the modes (3.25) and (3.26) which are the mixture of sound and usual Alfvén waves, become chiral when the anomalous transport taking into account as well.

In summary, we observe that CAWs may propagate in a chiral fluid with both left and right handed chiralities. Although the velocity of CAW in the chiral fluid with single right-handed chirality depends on the coefficient of gravitational anomaly, in QCD fluid its velocity depends on the coefficient of chiral anomaly. In addition,

¹³The two horizontal lines are related to the $\omega_{3,4}$ modes when $n = 0$ as well. At this limit, $\omega_{3,4}$ are nothing but the ordinary sound modes while $\omega_{5,6}$ vanish.

in the latter case the fluid must have finite density for propagating the CAWs, while in the former, CAW may even propagate at zero density [34].

4 Rotating QCD Fluid

In this part we consider a QCD fluid, rotating with constant vorticity $\boldsymbol{\Omega}$ in the absence of electromagnetic fields with the four velocity

$$u^\mu = \gamma \left(1, \boldsymbol{\Omega} \times \boldsymbol{x} \right). \quad (4.1)$$

In what follows, we consider the regime $\Omega r \ll 1$, where r is the distance from the axis of the vorticity. In this regime the Lorentz factor may be expanded as $\gamma = 1 + O\left((\Omega r)^2\right)$, so the vorticity (2.4) computed in equilibrium becomes

$$\omega^\mu = \left(0, \boldsymbol{\Omega} \right). \quad (4.2)$$

In the following subsections, after linearizing the equations around equilibrium state with the above vorticity, we compute the hydrodynamic modes and specify each of them carry which set of perturbations.

4.1 Equations of Motion Linearized

Let us take the small deviation of hydrodynamic fields (2.9) away from their equilibrium values as the following

$$\delta\phi_a = \left(\delta T, \boldsymbol{\pi}, \delta\mu, \delta\mu_5 \right). \quad (4.3)$$

In order to linearize the equations of motion, we have to expand the equations (2.1) around the equilibrium state:

$$\begin{aligned} u^\mu &= \left(1, \boldsymbol{\Omega} \times \boldsymbol{x} \right) \quad \Omega r \ll 1, \\ T &= \text{Const.}, \quad \mu = \text{Const.}, \quad \mu_5 = \text{Const.} \end{aligned} \quad (4.4)$$

and keep the terms up to first order in $\delta\phi_a$ fields. Among all terms, there is a delicate point regarding the expansion of vorticity terms around equilibrium in (2.4) and (2.5) which deserves to be explained in detail. Consider the velocity of fluid is perturbed by $\delta u^\mu = (\delta u^0, \delta \boldsymbol{u})$ as

$$u^\mu + \delta u^\mu = \left(1 + \delta u^0, \boldsymbol{\Omega} \times \boldsymbol{x} + \delta \boldsymbol{u} \right). \quad (4.5)$$

Demanding the above velocity to satisfy the relativistic normalization $u^\mu u_\mu = -1$, it turns out that the zero component of the perturbation must be

$$\delta u^0 = \left(\boldsymbol{\Omega} \times \boldsymbol{x} \right) \cdot \delta \boldsymbol{u}. \quad (4.6)$$

So to first order in perturbations, the vorticity takes the following form

$$\omega^\mu + \delta\omega^\mu = \left(\frac{\xi}{w} (\delta\boldsymbol{\pi} \cdot \boldsymbol{\Omega}), \boldsymbol{\Omega} + \frac{1}{2w} \boldsymbol{\nabla} \times \delta\boldsymbol{\pi} \right), \quad \Omega r \ll 1. \quad (4.7)$$

Using the above expression, the linearized equations of motion may be covariantly written as

$$M_{ab}^\Omega(\mathbf{k}, \omega) \delta\phi_a(\mathbf{k}, \omega) = 0, \quad (4.8)$$

with M_{ab}^Ω given by:

$$\begin{bmatrix} -i\alpha_1\omega & ik_j & -i\alpha_2\omega & -i\alpha_3\omega \\ i\alpha_1 v_s^2 k^i & -i\omega\delta_j^i - \epsilon^i{}_{jl}\Omega^l & i\alpha_2 v_s^2 k^i & i\alpha_3 v_s^2 k^i \\ -i\beta_1\omega + \left(\frac{\partial\xi}{\partial T}\right) i\boldsymbol{\Omega} \cdot \mathbf{k} & \frac{\bar{n}}{w} ik_j - \frac{2\xi}{w} i\omega\Omega_j & -i\beta_2\omega + \left(\frac{\partial\xi}{\partial\mu}\right) i\boldsymbol{\Omega} \cdot \mathbf{k} & -i\beta_3\omega + \left(\frac{\partial\xi}{\partial\mu_5}\right) i\boldsymbol{\Omega} \cdot \mathbf{k} \\ -i\gamma_1\omega + \left(\frac{\partial\xi_5}{\partial T}\right) i\boldsymbol{\Omega} \cdot \mathbf{k} & \frac{\bar{n}_5}{w} ik_j - \frac{2\xi_5}{w} i\omega\Omega_j & -i\gamma_2\omega + \left(\frac{\partial\xi_5}{\partial\mu}\right) i\boldsymbol{\Omega} \cdot \mathbf{k} & -i\gamma_3\omega + \left(\frac{\partial\xi_5}{\partial\mu_5}\right) i\boldsymbol{\Omega} \cdot \mathbf{k} \end{bmatrix}.$$

At this moment, since all the components of the matrix M are non-vanishing, one may think that each of the characteristics of the fluid is a coherent perturbation of all six scalar and vector hydro variables. We will show in the following that in fact, two of the characteristics are scalar type while the other four are the mixed scalar-vector perturbations.

4.2 Hydro Modes

Computing the eigenvalues of matrix $M_{ab} + \omega \mathbf{1}_{ab}$, or equivalently the roots of $\det M = 0$, we find six independent hydrodynamic modes of the fluid. Two of the corresponding 6-dimensional eigenvectors have no components in the 3-dimensional subspace of the state-space, generated by the $\boldsymbol{\pi}$ fields. So these two eigenvectors identify the integral curves along which, those Riemann invariant propagate that are combinations of just the scalar hydro variables. We consider these eigenvectors and their associated characteristics as the scalar sector. It turns out that the other four modes constitute the scalar-vector sector.

4.2.1 Scalar Sector: Chiral Vortical Waves

The eigen vectors corresponding to the scalar modes are

$$\delta\tilde{\phi}_{1,2} = \left(r \frac{\alpha_2}{\alpha_1} + s \frac{\alpha_3}{\alpha_1}, \mathbf{0}, -r, -s \right), \quad i = 1, 2 \quad (4.9)$$

with r and s the arbitrary parameters. Note that we have the freedom to choose any two arbitrary vectors with the above form as the eigen vectors associated with the following characteristics:

$$\omega_{1,2}(k) = -\frac{\mathcal{A}_3 \pm \sqrt{\mathcal{A}_3^2 - \mathcal{E}\mathcal{A}_4}}{\mathcal{E}} \boldsymbol{\Omega} \cdot \mathbf{k} \quad (4.10)$$

where \mathcal{A}_3 and \mathcal{A}_Δ may be written as polynomials of anomaly coefficients:

$$\begin{aligned}\mathcal{A}_3 &= \mathcal{D}x_1 + \mathcal{C}x_2 \\ \mathcal{A}_4 &= \mathcal{C}^2 y_1 + \mathcal{D}^2 y_2 + \mathcal{C}\mathcal{D}y_3\end{aligned}\tag{4.11}$$

with

$$\begin{aligned}x_1 &= \frac{T}{2} \left(\alpha_{[2]\beta_3} + \frac{Tn_5}{w} \alpha_{[2]\beta_1} - \frac{Tn}{w} \alpha_{[2]\gamma_1} + \frac{n_5\mu_5}{w} \alpha_{[3]\beta_2} - \frac{n\mu_5}{w} \alpha_{[3]\gamma_2} + \frac{2\mu_5 T}{w} \right) \\ x_2 &= \frac{1}{2w} (n_5 \alpha_{[2]\beta_1} - n \alpha_{[2]\gamma_1}) (\mu^2 + \mu_5^2) + \frac{\mu\mu_5}{w} (n_5 \alpha_{[1]\beta_3} - n \alpha_{[1]\gamma_3}) \\ &\quad - \frac{\mu}{2} (\alpha_{[1]\beta_3} + \alpha_{[1]\gamma_2}) + \frac{\mu_5}{2} (\alpha_{[1]\beta_2} + \alpha_{[1]\gamma_3}) + \frac{\mu_5}{w} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E}\end{aligned}$$

and

$$\begin{aligned}y_1 &= \alpha_1 \mu^2 \left(1 - \frac{\mu n}{w} \right) - \alpha_1 \mu_5^2 \left(1 - \frac{\mu_5 n_5}{w} \right) + \frac{\alpha_1 \mu \mu_5}{w} (\mu_5 n - \mu n_5) \\ &\quad - \frac{(\alpha_{[1]\beta_3} + \alpha_{[1]\gamma_2}) \mu \mu_5}{w} \left(\mu^2 + \frac{\mu_5^2}{3} \right) + \frac{(\alpha_{[1]\beta_2} + \alpha_{[1]\gamma_3}) \mu_5^2}{w} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \\ &\quad + (\alpha_{[2]\beta_1} n_5 - \alpha_{[2]\gamma_1} n) \frac{4\mu^2 \mu_5^3}{3w^2} + 2(\alpha_{[1]\beta_3} n_5 - \alpha_{[1]\gamma_3} n) \frac{\mu \mu_5^2}{w^2} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \\ &\quad + (\alpha_{[2]\beta_1} n_5 - \alpha_{[2]\gamma_1} n) \frac{\mu_5}{w^2} \left(\mu^4 + \frac{\mu_5^4}{3} \right) + \frac{\mu_5^2}{w^2} \left(\mu^2 + \frac{\mu_5^2}{3} \right)^2 \mathcal{E}\end{aligned}\tag{4.12}$$

$$y_2 = T^3 \left(\frac{\alpha_2 n}{w} + (\alpha_{[1]\gamma_2} n - \alpha_{[1]\beta_2} n_5) \frac{\mu_5 T}{3w^2} + 2(\alpha_{[2]\gamma_3} n - \alpha_{[2]\beta_3} n_5) \frac{\mu_5^2}{w^2} + \alpha_{[2]\beta_3} \frac{\mu_5}{w} + \frac{\mu_5^2 T}{w^2} \mathcal{E} \right)\tag{4.13}$$

$$\begin{aligned}y_3 &= T \left((\alpha_3 \mu_5 - \alpha_2 \mu) \left(1 - \frac{2\mu_5 n_5}{w} \right) + \alpha_1 (\mu_5 n_5 - \mu n) \frac{T}{w} + \alpha_2 (\mu^2 - \mu_5^2) \frac{n}{w} \right. \\ &\quad + \left(\alpha_{[2]\beta_3} \frac{\mu_5}{w} + \frac{2\mu_5^2 n}{w^2} \alpha_{[2]\gamma_3} - \frac{2\mu_5^2 n_5}{w^2} \alpha_{[2]\beta_3} \right) \left(\mu^2 + \frac{\mu_5^2}{3} \right) + (\alpha_{[1]\gamma_2} n - \alpha_{[1]\beta_2} n_5) \frac{4\mu^3 T}{3w^2} \\ &\quad + \left((\alpha_{[1]\beta_3} \mu_5 - \alpha_{[1]\beta_2} \mu) n_5 + (\alpha_{[1]\gamma_2} \mu - \alpha_{[1]\gamma_3} \mu_5) n \right) \frac{2\mu \mu_5 T}{w^2} \\ &\quad \left. + \left((\alpha_{[1]\beta_2} \mu_5 - \alpha_{[1]\gamma_2} \mu) + (\alpha_{[1]\gamma_3} \mu_5 - \alpha_{[1]\beta_3} \mu) \right) \frac{\mu_5 T}{w} + \frac{2\mu_5^2 T}{w^2} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \right)\end{aligned}\tag{4.14}$$

These two modes which carry the perturbations of temperature together with the vector and axial chemical potentials, are in fact the generalizations of the Chiral Vortical Waves firstly found in [32]. As it can be clearly seen in (4.10), one of the CVWs moves faster than the other. Only in the special limit where the fluid is non-chiral, namely when $\mathcal{A}_3 = 0$, the velocities of two CVWs become the same, while definitely propagating in opposite directions [32].

4.2.2 Scalar-Vector Sector

In addition to the two scalar modes computed in previous subsection, there are yet four modes which has to be identified. Computing the eigenvectors of the matrix M_{ab}^Ω , we find for $i = 3, 4, 5, 6$:

$$\begin{aligned}\delta\tilde{\phi}_i &= \left(\delta T, \delta\boldsymbol{\pi}, \delta\mu, \delta\mu_5 \right) \\ &= \left(1, -\frac{w\mathcal{E}\omega_i\mathbf{k}}{C_2k^2} + \frac{w\mathcal{E}\left(i\omega_i^2(\boldsymbol{\Omega}\times\mathbf{k}) + \omega_i(\boldsymbol{\Omega}\cdot\mathbf{k})\boldsymbol{\Omega} - \omega_i(\boldsymbol{\Omega}\cdot\hat{\mathbf{k}})^2\mathbf{k}\right)}{C_2(\omega_i^2k^2 - (\boldsymbol{\Omega}\cdot\mathbf{k})^2)}, \frac{C_1}{C_2}, \frac{C_3}{C_2} \right)\end{aligned}\quad (4.15)$$

with C_1 , C_2 and C_3 given in (3.22), (3.23) and (3.24). In the expression (4.15), ω_i is the dispersion relation of i^{th} mode ($i = 3, 4, 5, 6$)

$$\omega_{3,4} = \pm \frac{1}{\sqrt{2}} \sqrt{c_s^2k^2 + \Omega^2 + \sqrt{(c_s^2k^2 + \Omega^2)^2 - 4c_s^2k^2\Omega^2\cos^2\theta}} \quad (4.16)$$

$$\omega_{5,6} = \pm \frac{1}{\sqrt{2}} \sqrt{c_s^2k^2 + \Omega^2 - \sqrt{(c_s^2k^2 + \Omega^2)^2 - 4c_s^2k^2\Omega^2\cos^2\theta}}. \quad (4.17)$$

In contrast to the CVWs, the above four modes are independent of the axial chemical potential. This is simply due to this fact that in the absence of magnetic field, the vector and axial currents do not couple to the energy and momentum currents. However, to clarify the difference between $\omega_{3,4}$ and $\omega_{5,6}$ modes in this limit, let us now consider two special cases in the following:

• $\boldsymbol{\Omega} \parallel \mathbf{k}$

Consider $\mathbf{k} = (0, 0, k)$, in this case the dispersion relations and corresponding eigenvectors of the four modes above may be written as the following:

$$\begin{aligned}\omega_{3,4} &= \pm c_s k, & \delta\tilde{\phi}_a &= \left(1, 0, 0, \mp \frac{w\mathcal{E}}{C_2} c_s, \frac{C_1}{C_2}, \frac{C_3}{C_2} \right) \\ \omega_{5,6} &= \pm \Omega, & \delta\tilde{\phi}_a &= (0, 1, \pm i, 0, 0, 0)\end{aligned}\quad (4.18)$$

It is obvious that the modes $\omega_{3,4}$ are longitudinal left- and right-moving sound modes, while the modes $\omega_{5,6}$ are two oppositely polarized non-propagating vortices.

• $\boldsymbol{\Omega} \perp \mathbf{k}$

In this case the modes $\omega_{5,6}$ are not excited and for the other two modes we have:

$$\omega_{3,4} = \pm \sqrt{c_s^2k^2 + \Omega^2} k, \quad \delta\tilde{\phi}_a = \left(1, \frac{w\mathcal{E}}{C_2k^2} \left(i\boldsymbol{\Omega} \times \mathbf{k} \mp \sqrt{c_s^2k^2 + \Omega^2} \mathbf{k} \right), \frac{C_1}{C_2}, \frac{C_3}{C_2} \right) \quad (4.19)$$

These modes which are mixed of longitudinal and transverse propagation, are the modified sound modes. The velocity of these modes

$$v = \pm \frac{c_s^2 k}{\sqrt{c_s^2k^2 + \Omega^2}} \quad (4.20)$$

is greater than the ordinary sound velocity and this is simply due to the extra pressure originated from the acting of the Coriolis force on the transverse momentum perturbations. The situation is somewhat similar to the propagation of magnetosonic waves perpendicular to the magnetic field.

In general when the wave vector is neither parallel nor transverse to the vorticity, the four modes become mixed Sound-Coriolis modes which also disperse when propagate.

5 Rotating QCD Fluid Coupled to Magnetic Field

In this section we consider the general case in which the QCD fluid is either rotating and coupled to an external magnetic field. The associated results are lengthy and complicated, so we just mention the hydrodynamic modes in this case. However, in the following, we find the simplified expressions for the case of heavy ion plasma.

5.1 Equations of Motion Linearized

The thermodynamic equilibrium state of the fluid may be given by

$$\begin{aligned} u^\mu &= \left(1, \boldsymbol{\Omega} \times \boldsymbol{x}\right) & \Omega r \ll 1, \\ T &= \text{Const.}, \quad \mu = \text{Const.}, \quad \mu_5 = \text{Const.} \\ \boldsymbol{B} &= \text{Const.} \end{aligned} \tag{5.1}$$

If we slightly perturb the above state as

$$\phi_a + \delta\phi_a = \left(T + \delta T, \quad \mathbf{0} + \boldsymbol{\pi}, \quad \mu + \delta\mu, \quad \mu_5 + \delta\mu_5\right), \tag{5.2}$$

the linearized equations of motion take the following form

$$M_{ab}^{B\Omega}(\mathbf{k}, \omega) \delta\phi_a(\mathbf{k}, \omega) = 0, \tag{5.3}$$

with $M_{ab}^{B\Omega}$ given by (5.4). The interesting point with this matrix is the appearance of the underlined coupling terms between the vorticity and the magnetic field. Although, this coupling disappears in the case of quark gluon plasma wherein the magnetic field is parallel to the vorticity.

As in the previous two sections, two scalar modes together with four mixed scalar-vector modes constitute the full spectrum of the hydrodynamic excitations. As we will see, the scalar sector include the mixed CMWVs, while in the scalar-vector sector

there exist mixed Sound-Alfvén-Coriolis waves.

$$\begin{bmatrix}
-i\alpha_1\omega & ik_j & -i\alpha_2\omega & -i\alpha_3\omega \\
i\alpha_1 v_s^2 k^i & -i\omega\delta_j^i - \epsilon^i{}_{jl}\Omega^l - \frac{\bar{n}}{w}\epsilon^i{}_{jl}B^l & i\alpha_2 v_s^2 k^i & i\alpha_3 v_s^2 k^i \\
& -i\frac{\xi}{2w}(\mathbf{B} \cdot \mathbf{k}\delta_j^i - B_j k^i) & + \left(\frac{\partial\xi}{\partial\mu}\right)\underline{(\mathbf{B} \times \boldsymbol{\Omega})}^i & + \left(\frac{\partial\xi}{\partial\mu_5}\right)\underline{(\mathbf{B} \times \boldsymbol{\Omega})}^i \\
-i\beta_1\omega + \left(\frac{\partial\xi}{\partial T}\right)i\boldsymbol{\Omega} \cdot \mathbf{k} & \frac{\bar{n}}{w}ik_j - \frac{2\xi}{w}i\omega\Omega_j & -i\beta_2\omega + \left(\frac{\partial\xi}{\partial\mu}\right)i\boldsymbol{\Omega} \cdot \mathbf{k} & -i\beta_3\omega + \left(\frac{\partial\xi}{\partial\mu_5}\right)i\boldsymbol{\Omega} \cdot \mathbf{k} \\
+ \left(\frac{\partial\xi_B}{\partial T}\right)i\mathbf{B} \cdot \mathbf{k} & -\frac{\xi_B}{w}i\omega B_j - \frac{\xi_B}{w}\underline{(\mathbf{B} \times \boldsymbol{\Omega})}_j & + \left(\frac{\partial\xi_B}{\partial\mu}\right)i\mathbf{B} \cdot \mathbf{k} & + \left(\frac{\partial\xi_B}{\partial\mu_5}\right)i\mathbf{B} \cdot \mathbf{k} \\
-i\gamma_1\omega + \left(\frac{\partial\xi_5}{\partial T}\right)i\boldsymbol{\Omega} \cdot \mathbf{k} & \frac{\bar{n}_5}{w}ik_j - \frac{2\xi_5}{w}i\omega\Omega_j & -i\gamma_2\omega + \left(\frac{\partial\xi_5}{\partial\mu}\right)i\boldsymbol{\Omega} \cdot \mathbf{k} & -i\gamma_3\omega + \left(\frac{\partial\xi_5}{\partial\mu_5}\right)i\boldsymbol{\Omega} \cdot \mathbf{k} \\
+ \left(\frac{\partial\xi_{5B}}{\partial T}\right)i\mathbf{B} \cdot \mathbf{k} & -\frac{\xi_{5B}}{w}i\omega B_j - \frac{\xi_{5B}}{w}\underline{(\mathbf{B} \times \boldsymbol{\Omega})}_j & + \left(\frac{\partial\xi_{5B}}{\partial\mu}\right)i\mathbf{B} \cdot \mathbf{k} & + \left(\frac{\partial\xi_{5B}}{\partial\mu_5}\right)i\mathbf{B} \cdot \mathbf{k}
\end{bmatrix} \quad (5.4)$$

5.2 Hydro Modes

The dispersion relations of the two scalar modes, namely the CMVWs in this case may be given as

$$\begin{aligned}
\omega_{1,2} = & -\frac{1}{\mathcal{E}} \left(\mathcal{A}_1 \mathbf{B} \cdot \mathbf{k} + \mathcal{A}_3 \boldsymbol{\Omega} \cdot \mathbf{k} \right) \\
& \pm \frac{1}{\mathcal{E}} \sqrt{\left(\mathcal{A}_1 \mathbf{B} \cdot \mathbf{k} + \mathcal{A}_3 \boldsymbol{\Omega} \cdot \mathbf{k} \right)^2 - \mathcal{E} \left(\mathcal{A}_1 (\mathbf{B} \cdot \mathbf{k})^2 + \mathcal{A}_5 \mathbf{B} \cdot \mathbf{k} \boldsymbol{\Omega} \cdot \mathbf{k} + \mathcal{A}_4 (\boldsymbol{\Omega} \cdot \mathbf{k})^2 \right)}
\end{aligned} \quad (5.5)$$

with the new anomaly expression:

$$\begin{aligned}
\mathcal{A}_5 = & \mathcal{C} \left\{ (3\mu^2\mu_5^2 + \frac{1}{3}\mu_5^4)(n_5\alpha_{[1}\beta_3] - n\alpha_{[1}\gamma_3]) - (\frac{4}{3}\mu\mu_5^3 + 2\mu^3\mu_5)(n_5\alpha_{[1}\beta_2] - n\alpha_{[1}\gamma_2]) \right. \\
& - w(\mu^2\mu_5 + \frac{1}{3}\mu_5^3)(\alpha_{[1}\beta_3] + \alpha_{[1}\gamma_2]) + \mu\mu_5^2w(\alpha_{[1}\beta_2] + \alpha_{[1}\gamma_3]) + 2\mu\mu_5^2 \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \\
& \left. - 2\alpha_1\mu w(\mu n + \mu_5 n_5 - w) \right\} + \mathcal{CDT} \left\{ -2\mu\mu_5^2(n_5\alpha_{[2}\beta_3] - n\alpha_{[2}\gamma_3]) + \mu\mu_5 w\alpha_{[2}\gamma_3] \right. \\
& + \mu_5^2(n_5\alpha_{[1}\beta_3] - n\alpha_{[1}\gamma_3]) - \mu_5 T w(\alpha_{[1}\beta_3] + \alpha_{[1}\gamma_2]) - 2\mu\mu_5 n_5 T(n_5\alpha_{[1}\beta_2] - n\alpha_{[1}\gamma_2]) \\
& \left. - (\alpha_1 n T - \alpha_2(\mu n + 2\mu_5 n_5 - w) - \alpha_3 \mu_5 n) w + 2\mu\mu_5^2 T \mathcal{E} \right\}
\end{aligned} \quad (5.6)$$

These are in fact two waves with different velocities. In the non-chiral limit where \mathcal{A}_1 and \mathcal{A}_3 vanish, the velocities of two modes become the same.

In case of the scalar-vector modes, the dispersion relations are so complicated. We first give the dispersion relation of each mode at zero order of hydrodynamic constitutive currents:

$$\tilde{\omega}_{3,4,5,6} = \frac{\pm 1}{\sqrt{2}} \sqrt{\left(\mathbf{B}_w^n + \boldsymbol{\Omega}\right)^2 + c_s^2 k^2 \pm \sqrt{\left(\left(\mathbf{B}_w^n + \boldsymbol{\Omega}\right)^2 + c_s^2 k^2\right)^2 - 4c_s^2 \left(\mathbf{k} \cdot \mathbf{B}_w^n + \mathbf{k} \cdot \boldsymbol{\Omega}\right)^2}}. \quad (5.7)$$

By use of the above four zero order expressions, one may write the dispersion relations up to first order for $i = 3, 4, 5, 6$:

$$\omega_i = \tilde{\omega}_i - \frac{1}{\mathcal{E}} \frac{(\Sigma_j a_j \mathbf{s}_j) \tilde{\omega}_i^4 + (i b \mathbf{s}_7) \tilde{\omega}_i^3 + (\Sigma_{j,k} c_{j,k} \mathbf{s}_j \mathbf{s}_k) \tilde{\omega}_i^2 + \Sigma_{j,k,l} c_{j,k,l} \mathbf{s}_j \mathbf{s}_k \mathbf{s}_l}{3 \tilde{\omega}_i^4 - 2 \left(k^2 c_s^2 + \left(\boldsymbol{\Omega} + \frac{\bar{n}}{w} \mathbf{B}\right)^2\right) \tilde{\omega}_i^2 + c_s^2 \left(\boldsymbol{\Omega} \cdot \mathbf{k} + \frac{\bar{n}}{w} \mathbf{B} \cdot \mathbf{k}\right)^2} \quad (5.8)$$

with

$$\mathcal{E} = -\epsilon^{ijk} \alpha_i \beta_j \gamma_k \quad (\epsilon^{123} = 1). \quad (5.9)$$

In the equation (5.8), $\{\mathbf{s}_i\}$ is the set of scalars made out of three independent vectors \mathbf{k} , \mathbf{B} and $\boldsymbol{\Omega}$

$$\begin{aligned} \mathbf{s}_1 &= \mathbf{k}^2, \\ \mathbf{s}_2 &= \mathbf{B}^2, & \mathbf{s}_3 &= \mathbf{B} \cdot \mathbf{k}, \\ \mathbf{s}_4 &= \boldsymbol{\Omega}^2, & \mathbf{s}_5 &= \boldsymbol{\Omega} \cdot \mathbf{k}, \\ \mathbf{s}_6 &= \mathbf{B} \cdot \boldsymbol{\Omega}, & \mathbf{s}_7 &= \mathbf{k} \cdot \mathbf{B} \times \boldsymbol{\Omega}. \end{aligned} \quad (5.10)$$

We have also defined a scalar b together with a vector a_j and two tensors $c_{j,k}$ and $d_{j,k,l}$ in the seven-dimensional space generated by the above scalars. All these objects are in terms of the components of the susceptibility matrix and the anomaly coefficients. The scalar b is given by

$$\begin{aligned} b = & \mathcal{C} \left\{ \frac{1}{2} (\mu_5 \beta_{[1} \gamma_{3]} - \mu \beta_{[1} \gamma_{2]}) + \frac{n}{2w} \left(\beta_{[1} \gamma_{2]} - \frac{n}{w} \alpha_{[1} \gamma_{2]} + \frac{n_5}{w} \alpha_{[1} \beta_{2]} \right) (\mu^2 + \mu_5^2) \right. \\ & + \frac{\mu_5}{w} \left(\frac{\mu n}{w} - \frac{1}{2} \right) (n \alpha_{[1} \gamma_{3]} - n_5 \alpha_{[1} \beta_{3]}) + \frac{\mu}{2w} (n \alpha_{[1} \gamma_{2]} - n_5 \alpha_{[1} \beta_{2]}) \\ & \left. - \frac{\mu \mu_5 n}{w} \beta_{[1} \gamma_{3]} + \frac{\mu_5 n}{2w^2} \left(c_s^2 - \frac{1}{2} \right) \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} + \frac{\mu \mu_5}{4w} \mathcal{E} \right\} \\ & \mathcal{DT} \left\{ \frac{nT}{2w} \left(\beta_{[1} \gamma_{2]} - \frac{n}{w} \alpha_{[1} \gamma_{2]} + \frac{n_5}{w} \alpha_{[1} \beta_{2]} \right) \right. \\ & \left. + \frac{n \mu_5}{w} \left(\beta_{[2} \gamma_{3]} - \frac{n}{w} \alpha_{[2} \gamma_{3]} + \frac{n_5}{w} \alpha_{[2} \beta_{3]} \right) + \frac{\mu_5 n T}{2w^2} \left(c_s^2 - \frac{1}{2} \right) \mathcal{E} \right\} \end{aligned} \quad (5.11)$$

and the non-vanishing components of the vector a_j are

$$a_3 = \mathcal{C} \left\{ \frac{1}{2} (\alpha_{[3}\beta_{1]} + \alpha_{[2}\gamma_{1]} + \left(\frac{\mu_5 n_5}{2w} \alpha_{[1}\beta_{3]} - \frac{\mu n_5}{2w} \alpha_{[1}\beta_{2]} + \frac{\mu n}{2w} \alpha_{[1}\gamma_{2]} - \frac{\mu_5 n}{2w} \alpha_{[1}\gamma_{3]} \right) \right. \\ \left. - \frac{\mu_5 n}{2w^2} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} + \frac{3\mu\mu_5}{2w} \mathcal{E} \right\} - \mathcal{D} \left\{ \frac{\mu_5 n T^2}{2w^2} \right\} \mathcal{E} \quad (5.12)$$

$$a_5 = \mathcal{C} \left\{ \frac{1}{2} (-\mu \alpha_{[1}\beta_{3]} + \mu_5 \alpha_{[1}\beta_{2]} - \mu \alpha_{[1}\gamma_{2]} + \mu_5 \alpha_{[1}\gamma_{3]}) + \frac{\mu_5}{w} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \right. \\ \left. + \left(\alpha_{[1}\gamma_{2]} \frac{n}{2w} - \alpha_{[1}\beta_{2]} \frac{n_5}{2w} \right) (\mu^2 + \mu_5^2) + (\alpha_{[1}\beta_{3]} - \alpha_{[1}\gamma_{3]}) \frac{\mu\mu_5 n_5}{w} \right\} \\ + \mathcal{D} T \left\{ \frac{1}{2} \alpha_{[2}\beta_{3]} + \frac{\mu_5}{w} (\alpha_{[2}\gamma_{3]} n - \alpha_{[2}\beta_{3]} n_5) + \frac{T}{2w} (n \alpha_{[1}\gamma_{2]} - n_5 \alpha_{[1}\beta_{2]}) + \frac{\mu_5 T}{w} \mathcal{E} \right\} \quad (5.13)$$

The tensor $c_{j,k}$ is a symmetric tensor with the following non-vanishing components:

$$c_{5,2} = c_{2,5} = \mathcal{C} \left\{ \frac{n}{2w^2} ((\mu n - \mu_5 n_5) \alpha_{[1}\beta_{3]} + (\mu n_5 - \mu_5 n) \alpha_{[1}\beta_{2]}) \right. \\ \left. - \frac{n^2}{2w^2} \beta_{[1}\gamma_{2]} (\mu^2 + \mu_5^2) + \frac{\mu n}{2w} \beta_{[1}\gamma_{2]} - \frac{\mu_5 n}{2w} \left(1 - \frac{2\mu n}{w} \right) \beta_{[1}\gamma_{3]} \right. \\ \left. - \frac{\mu_5 n^2}{w^3} \left(\frac{c_s^2}{2} + \frac{5}{4} \right) \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} + \frac{\mu\mu_5 n}{4w^2} \mathcal{E} \right\} \quad (5.14) \\ + \mathcal{D} T \left\{ -\frac{n^2}{2w^2} (2\mu_5 \beta_{[2}\gamma_{3]} + \alpha_{[2}\beta_{3]} + T \beta_{[1}\gamma_{2]}) - \frac{\mu_5 n^2 T}{w^3} \left(\frac{c_s^2}{2} + \frac{5}{2} \right) \mathcal{E} \right\}$$

$$c_{5,6} = c_{6,5} = \mathcal{C} \left\{ \frac{1}{2} (\mu \beta_{[1}\gamma_{2]} - \mu_5 \beta_{[1}\gamma_{3]}) - \frac{n}{2w} \left(\beta_{[1}\gamma_{2]} + \frac{n}{w} \alpha_{[1}\gamma_{2]} - \frac{n_5}{w} \alpha_{[1}\beta_{2]} \right) (\mu^2 + \mu_5^2) \right. \\ \left. + \frac{\mu\mu_5 n}{w} \left(\beta_{[1}\gamma_{3]} + \frac{n}{w} \alpha_{[1}\gamma_{3]} - \frac{n_5}{w} \alpha_{[1}\beta_{3]} \right) + \frac{n}{w} (\mu \alpha_{[1}\beta_{3]} - \mu_5 \alpha_{[1}\beta_{2]}) + \frac{\mu\mu_5}{4w} \mathcal{E} \right. \\ \left. + \frac{1}{2w} (\mu n \alpha_{[1}\gamma_{2]} - \mu_5 n \alpha_{[1}\gamma_{3]} + \mu n_5 \alpha_{[1}\beta_{2]} - \mu_5 n_5 \alpha_{[1}\beta_{3]}) - \frac{\mu_5 n}{w^2} \left(\frac{c_s^2}{2} + \frac{9}{4} \right) \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \right\} \\ + \mathcal{D} T \left\{ -\frac{n}{w} \alpha_{[2}\beta_{3]} - \frac{nT}{2w} \left(\beta_{[1}\gamma_{2]} + \frac{n}{w} \alpha_{[1}\gamma_{2]} - \frac{n_5}{w} \alpha_{[1}\beta_{2]} \right) \right. \\ \left. - \frac{\mu_5 n}{w} \left(\beta_{[2}\gamma_{3]} + \frac{n}{w} \alpha_{[2}\gamma_{3]} - \frac{n_5}{w} \alpha_{[2}\beta_{3]} \right) - \frac{\mu_5 n T}{w^2} \left(\frac{c_s^2}{2} + \frac{9}{4} \right) \mathcal{E} \right\} \quad (5.15)$$

$$\begin{aligned}
c_{5,4} = c_{4,5} = & \mathcal{C} \left\{ \frac{\mu}{2} (\alpha_{[1]\beta_3} + \alpha_{[1]\gamma_2}) - \frac{\mu_5}{2} (\alpha_{[1]\beta_2} + \alpha_{[1]\gamma_3}) \right\} \\
& + \mathcal{DT} \left\{ \frac{1}{2w} (n_5 \alpha_{[1]\beta_2} - n \alpha_{[1]\gamma_2}) (\mu^2 + \mu_5^2) - \frac{\mu\mu_5}{w} (n_5 \alpha_{[1]\beta_3} - n \alpha_{[1]\gamma_3}) - \frac{\mu_5}{w} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \right\}
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
c_{3,6} = c_{6,3} = & \mathcal{C} \left\{ \frac{n^2}{2w^2} \left(\beta_{[1]\gamma_2} - \frac{n}{w} \alpha_{[1]\gamma_2} + \frac{n_5}{w} \alpha_{[1]\beta_2} \right) (\mu^2 + \mu_5^2) - \frac{9\mu\mu_5 n}{4w^2} \mathcal{E} \right. \\
& + \frac{n}{2w^2} (\mu n \alpha_{[1]\gamma_2} - \mu_5 n \alpha_{[1]\gamma_3} + \mu n_5 \alpha_{[1]\beta_2} - \mu_5 n_5 \alpha_{[1]\beta_3}) \\
& - \frac{\mu\mu_5 n^2}{w^2} \left(\beta_{[1]\gamma_3} - \frac{n}{w} \alpha_{[1]\gamma_3} + \frac{n_5}{w} \alpha_{[1]\beta_3} \right) + \frac{n}{w} (\alpha_{[1]\beta_3} + \alpha_{[1]\gamma_2}) \\
& \left. + \frac{n}{2w} (\mu_5 \beta_{[1]\gamma_3} - \mu \beta_{[1]\gamma_2}) + \frac{\mu_5 n^2}{2w^3} \left(c_s^2 + \frac{1}{2} \right) \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \right\} \\
& + \mathcal{DT} \left\{ \frac{\mu_5 n^2}{w^2} \left(\beta_{[2]\gamma_3} - \frac{n}{w} \alpha_{[2]\gamma_3} + \frac{n_5}{w} \alpha_{[2]\beta_3} \right) \right. \\
& \left. + \frac{n^2 T}{2w^2} \left(\beta_{[1]\gamma_2} - \frac{n}{w} \alpha_{[1]\gamma_2} + \frac{n_5}{w} \alpha_{[1]\beta_2} \right) + \frac{\mu_5 n^2 T^2}{2w^3} \left(c_s^2 + \frac{1}{2} \right) \mathcal{E} \right\}
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
c_{3,4} = c_{4,3} = & \mathcal{C} \left\{ \frac{1}{2} (\alpha_{[1]\beta_3} + \alpha_{[1]\gamma_2}) + \frac{1}{2} (\mu_5 \beta_{[1]\gamma_3} - \mu \beta_{[1]\gamma_2}) - \frac{5\mu\mu_5}{4w} \mathcal{E} \right. \\
& + \frac{n}{2w} \left(\beta_{[1]\gamma_2} - \frac{n}{w} \alpha_{[1]\gamma_2} + \frac{n_5}{w} \alpha_{[1]\beta_2} \right) - \frac{\mu\mu_5 n}{w} \left(\beta_{[1]\gamma_3} - \frac{n}{w} \alpha_{[1]\gamma_3} + \frac{n_5}{w} \alpha_{[1]\beta_3} \right) \\
& + \frac{\mu_5 n}{w^2} \left(\frac{c_s^2}{2} + \frac{1}{4} \right) \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \left. \right\} + \mathcal{DT} \left\{ \frac{nT}{2w} \left(\beta_{[1]\gamma_2} - \frac{n}{w} \alpha_{[1]\gamma_2} + \frac{n_5}{w} \alpha_{[1]\beta_2} \right) \right. \\
& \left. + \frac{\mu_5 n}{w} \left(\beta_{[2]\gamma_3} - \frac{n}{w} \alpha_{[2]\gamma_3} + \frac{n_5}{w} \alpha_{[2]\beta_3} \right) + \frac{\mu_5 n T}{w^2} \left(\frac{c_s^2}{2} + \frac{1}{4} \right) \mathcal{E} \right\}
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
c_{2,3} = c_{3,2} = & \mathcal{C} \left\{ \frac{n^2}{2w^2} (\alpha_{[1]\beta_3} + \alpha_{[1]\gamma_2}) - \frac{\mu\mu_5 n^2}{w^3} \mathcal{E} \right. \\
& \left. - \frac{n^2}{2w^3} (\mu n \alpha_{[1]\gamma_2} - \mu_5 n \alpha_{[1]\gamma_3} - \mu n_5 \alpha_{[1]\beta_2} + \mu_5 n_5 \alpha_{[1]\beta_3}) \right\}
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
c_{3,1} = c_{1,3} = & \mathcal{C} \left\{ \frac{c_s^2}{2} (\alpha_{[1]\beta_3} + \alpha_{[1]\gamma_2}) - \frac{c_s^2}{2w} (\mu n \alpha_{[1]\gamma_2} - \mu_5 n \alpha_{[1]\gamma_3} - \mu n_5 \alpha_{[1]\beta_2} + \mu_5 n_5 \alpha_{[1]\beta_3}) \right. \\
& \left. + \frac{c_s^2 \mu_5 n}{2w^2} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} - \frac{3c_s^2 \mu \mu_5}{2w} \mathcal{E} \right\} + \mathcal{DT} \left\{ \frac{c_s^2 \mu_5 n T^2}{2w^2} \mathcal{E} \right\}
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
c_{5,1} = c_{1,5} = & \mathcal{C} \left\{ \frac{\mu c_s^2}{2} (\alpha_{[1}\beta_3] + \alpha_{[1}\gamma_2]) - \frac{\mu_5 c_s^2}{2} (\alpha_{[1}\beta_2] + \alpha_{[1}\gamma_3]) - \frac{c_s^2 \mu_5}{w} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \right. \\
& \left. + \frac{c_s^2}{2w} (n_5 \alpha_{[1}\beta_2] - n \alpha_{[1}\gamma_2]) (\mu^2 + \mu_5^2) - \frac{c_s^2 \mu \mu_5}{w} (n_5 \alpha_{[1}\beta_3] - n \alpha_{[1}\gamma_3]) \right\} \\
& + \mathcal{DT} \left\{ \frac{c_s^2 T}{2w} (n_5 \alpha_{[1}\beta_2] - n \alpha_{[1}\gamma_2]) + \frac{c_s^2 \mu_5}{w} (n_5 \alpha_{[2}\beta_3] - n \alpha_{[2}\gamma_3]) - \frac{c_s^2}{2} \alpha_{[2}\beta_3] - \frac{c_s^2 \mu_5 T}{w} \mathcal{E} \right\}
\end{aligned} \tag{5.21}$$

The tensor $d_{j,k,l}$ is a fully symmetric rank-3 tensor with non-vanishing components:

$$\begin{aligned}
d_{3,3,3} = & \mathcal{C} \left\{ \frac{c_s^2 n^2}{2w^3} (\mu n \alpha_{[1}\gamma_2] - \mu_5 n \alpha_{[1}\gamma_3] - \mu n_5 \alpha_{[1}\beta_2] + \mu_5 n_5 \alpha_{[1}\beta_3]) \right. \\
& \left. - \frac{c_s^2 n^2}{2w^2} (\alpha_{[1}\beta_3] + \alpha_{[1}\gamma_2]) + \frac{c_s^2 \mu \mu_5 n^2}{w^3} \mathcal{E} \right\}
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
d_{5,5,5} = & \mathcal{C} \left\{ \frac{c_s^2 \mu_5}{2} (\alpha_{[1}\beta_2] + \alpha_{[1}\gamma_3]) - \frac{c_s^2 \mu}{2} (\alpha_{[1}\beta_3] + \alpha_{[1}\gamma_2]) + \frac{c_s^2 \mu_5}{w} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \right. \\
& \left. - \frac{c_s^2}{2w} (n_5 \alpha_{[1}\beta_2] - n \alpha_{[1}\gamma_2]) (\mu^2 + \mu_5^2) + \frac{c_s^2 \mu \mu_5}{w} (n_5 \alpha_{[1}\beta_3] - n \alpha_{[1}\gamma_3]) \right\} \\
& + \mathcal{DT} \left\{ -\frac{c_s^2 T}{2w} (n_5 \alpha_{[1}\beta_2] - n \alpha_{[1}\gamma_2]) - \frac{c_s^2 \mu_5}{w} (n_5 \alpha_{[2}\beta_3] - n \alpha_{[2}\gamma_3]) + \frac{c_s^2}{2} \alpha_{[2}\beta_3] + \frac{c_s^2 \mu_5 T}{w} \mathcal{E} \right\}
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
d_{3,3,5} = & \mathcal{C} \left\{ -\frac{c_s^2 n^2}{2w^3} (n_5 \alpha_{[1}\beta_2] - n \alpha_{[1}\gamma_2]) (\mu^2 + \mu_5^2) + \frac{c_s^2 \mu \mu_5 n^2}{w^3} (n_5 \alpha_{[1}\beta_3] - n \alpha_{[1}\gamma_3]) \right. \\
& + \frac{c_s^2 n^2}{2w^2} \left(\mu_5 (\alpha_{[1}\beta_2] - \alpha_{[1}\gamma_3]) - \mu (\alpha_{[1}\beta_3] - \alpha_{[1}\gamma_2]) \right) + \frac{c_s^2 \mu_5 n^2}{w^3} \left(\mu^2 + \frac{\mu_5^2}{3} \right) \mathcal{E} \\
& \left. + \frac{c_s^2 n_5 n}{w^2} (\mu_5 \alpha_{[1}\beta_3] - \mu \alpha_{[1}\beta_2]) - \frac{c_s^2 n}{w} (\alpha_{[1}\beta_3] + \alpha_{[1}\gamma_2]) + \frac{2c_s^2 \mu \mu_5 n}{w^2} \mathcal{E} \right\} \\
& - \mathcal{D} T \left\{ \frac{c_s^2 n^2 T}{2w^3} \left((n_5 \alpha_{[1}\beta_2] - n \alpha_{[1}\gamma_2]) + 2\mu_5 (n_5 \alpha_{[2}\beta_3] - n \alpha_{[2}\gamma_3]) \right) \right. \\
& \left. - \frac{c_s^2 n^2}{2w^2} \alpha_{[2}\beta_3] - \frac{c_s^2 \mu_5 n^2 T^2}{w^3} \mathcal{E} \right\}, (d_{3,3,5} = d_{3,5,3} = d_{5,3,3})
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
d_{3,5,5} = & + \mathcal{DT} \left\{ -\frac{c_s^2 n^2 T}{2w^3} (n_5 \alpha_{[1}\beta_2] - n \alpha_{[1}\gamma_2]) - \frac{c_s^2 \mu_5 n^2 T}{w^3} (n_5 \alpha_{[2}\beta_3] - n \alpha_{[2}\gamma_3]) \right. \\
& \left. + \frac{c_s^2 n^2}{2w^2} \alpha_{[2}\beta_3] + \frac{c_s^2 \mu_5 n^2 T^2}{w^3} \mathcal{E} \right\}, (d_{3,5,5} = d_{5,3,5} = d_{5,5,3})
\end{aligned} \tag{5.25}$$

Let us emphasize that due to difficulties in working with the equation (5.8), from now on we will focus on the special case wherein the magnetic field is parallel to the vorticity. This case might be more relevant to the QCD fluid produced in heavy ion collisions.

5.3 Hydro Modes in Special Case $\mathbf{k} \parallel \mathbf{B} \parallel \boldsymbol{\Omega}$

The structure of modes in this case is as it follows

$$\omega_{1,2} = -\frac{\mathcal{A}_1 B + \mathcal{A}_3 \Omega}{\mathcal{E}} k \quad (5.26)$$

$$\pm \frac{\sqrt{(\mathcal{A}_1 B + \mathcal{A}_3 \Omega)^2 - \mathcal{E}(\mathcal{A}_1 B^2 + \mathcal{A}_5 B \Omega + \mathcal{A}_4 \Omega^2)}}{\mathcal{E}} k$$

$$\omega_{3,4} = \pm \left(\frac{n}{w} B + \Omega \right) - \left\{ \mathcal{C} \left(\mu \mu_5 - \frac{n \mu_5}{3w} (3\mu^2 + \mu_5^2) \right) - \mathcal{D} \frac{n \mu_5}{w} T^2 \right\} \frac{B}{w} k \quad (5.27)$$

$$\omega_{5,6} = \pm k v_s. \quad (5.28)$$

What we observe in the scalar sector is the existence of two CMVWs, one propagating faster than the other. In the scalar-vector sector however, we encounter with an interesting separation between the sound modes and the CAWs. The most important result is the propagation of pure CAWs in this case. The propagation of the latter waves is also accompanied with two oppositely polarizing vortices.

There is an important point considering the first two modes given above, namely CMVWs. It is obvious that in general, the velocity of CMVW in (5.26) differs from the sum of the velocities of CMW (3.6) and CVW (4.10). However, in next subsection we will show that at zero axial density, $v_{cmvw} = v_{cmw} + v_{cvw}$.

5.4 Heavy Ion Collision

It has been understood that the quark gluon plasma produced in a heavy ion collision is non-chiral, i.e. $\mu_5 = 0$. In this, limit we have $\beta_3 = \alpha_3 = 0$ and the susceptibility matrix takes the following form:

$$\tilde{\chi} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{c_s^2 T} (w - \mu n) & \frac{n}{c_s^2} & 0 \\ \frac{1}{T} \left(\frac{n}{c_s^2} - \chi \mu \right) & \chi & 0 \\ 0 & 0 & \chi \end{bmatrix}.$$

As a result, the hydrodynamic modes given in (5.26), (5.27) and (5.28) become

$$\begin{aligned} \omega_{1,2} &= \pm (v_{CMW} + v_{CVW}) k \\ &= \pm \frac{Bk}{2\pi^2 \chi} \frac{1 - \frac{\mu n}{w}}{\sqrt{1 - \frac{\mu n}{w} - \frac{n}{\chi w} \left(\frac{n}{c_s^2} - \chi \mu \right)}} \pm \frac{\Omega \mu k}{2\pi^2 \chi} \frac{1 - \frac{\mu n}{w} - \frac{\pi^2 n T^2}{3w}}{\sqrt{1 - \frac{\mu n}{w} - \frac{n}{\chi w} \left(\frac{n}{c_s^2} - \chi \mu \right)}} \end{aligned} \quad (5.29)$$

$$\omega_{3,4} = \pm \left(\Omega_L + \Omega \right) \quad (5.30)$$

$$\omega_{5,6} = \pm k v_s. \quad (5.31)$$

Among the three equations given above, (5.29) is one of our main results in this paper, however before discussing further about that, we first consider two special cases below.

5.4.1 Comparison With Kharzeev-Yee Result [28] at Zero Density

As it is well-known the notion of CMW was firstly introduced by Kharzeev and Yee [28]. They argue that the combination of chiral magnetic effect with the chiral separation effect may produce a new kind of gapless excitations in a fluid with both axial and vector charges; this mode is called the Chiral Magnetic Wave. To obtain the dispersion relation of CMW, they keep the local energy and momentum in the fluid fixed and just allow the axial and vector densities to fluctuate. It is worth mentioning that in their computations, neither any background vector charge density nor the background axial charge density has been considered. As a result they find two excitations propagating either in the same and the opposite of the magnetic field both with the following velocity:

$$v_{KY} = \frac{B}{2\pi^2\chi} \quad (5.32)$$

where χ is the susceptibility.

In order to reproduce the above equation from our results, it is needed to get $\Omega = 0$ together with $\mu = 0$ and $\mu_5 = 0$ in (5.29). It is worth-mentioning that at $\mu = \mu_5 = 0$, the perturbations of the energy and momentum are fully decoupled of the vector and axial charge perturbations. Therefore, we can simply derive the result of Kharzeev-Yee just by taking the $\mu \rightarrow 0$ and $\mu_5 \rightarrow 0$ limits of our results. At this limit, it is not hard to see that $\omega_{1,2}$ in (5.29) gives exactly the left- and right-moving CMWs of [28]¹⁴.

In the above, we have reproduced the CMW of the [28] from our scalar sector modes. One has to recall that in this sector the temperature perturbations contribute as well, while the CMW in [28] is just the coherent perturbation of vector and axial currents. So it might be not clear why the temperature perturbations can not distinguish our result from that of [28]. A more careful look at the eigenvector (3.5) shows that in our present limit where $\alpha_2 = \alpha_3 = 0$, the temperature perturbations are indeed decoupled from the scalar sector. Let us emphasize that even if the vector chemical potential was non-vanishing ($\alpha_2 \neq 0$), the above discussion would no longer be valid and the temperature perturbations participated in the propagation of the CMW (see (3.20)) .

¹⁴Let us denote that when $\mu = \mu_5 = 0$, the $\tilde{\chi}$ simplifies to

$$\tilde{\chi} = \begin{bmatrix} c_v & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \chi \end{bmatrix}$$

5.4.2 CVW at Constant Temperature; Comparison With Jiang-Huang-Liao [32] Result

Analogous to the CMW in presence of a magnetic field, another gapless excitation may propagate when fluid is rotating. In this case, it is the combination of the chiral vortical effect with the chiral separation effect that leads to propagation of the Chiral Vortical Wave (CVW).

The velocity of the CVW was firstly computed by Jiang, Huang and Liao in [32]. Similar to Kharzeev-Yee, they allow just the vector and axial chemical potentials to fluctuate. Then at constant temperature $T = \text{Const.}$ and zero axial chemical potential $\mu_5 = 0$, they found the velocity of the CVW as

$$v_{CVW} = \frac{\mu \Omega}{2\pi^2 \chi}. \quad (5.33)$$

In contrast to the CMW, CVW does not propagate at zero vector charge density. It may be simply seen in the eigenvector (4.9); the temperature perturbations will be never decoupled from the scalar sector at finite density. Therefore, demanding the fluid to be at $T = \text{Const.}$ distinguishes between (5.33) and our result. As a result, our formula (5.29) will not coincide with the result of [32] in general.

In order to reproduce (5.33), we have to keep temperature and local momentum fixed, from the beginning in our computations. Doing so, the 6-dimensional state-space reduces to a 2-dimensional space, including the states characterized by either a vector chemical potential and an axial chemical potential. The matrix (5.4) so simplifies to

$$M_{T=\text{Const.}}^{B\Omega} = \begin{bmatrix} -i\beta_2\omega + \left(\frac{\partial \xi^T}{\partial \mu}\right) i\Omega \cdot \mathbf{k} & -i\beta_3\omega + \left(\frac{\partial \xi^T}{\partial \mu_5}\right) i\Omega \cdot \mathbf{k} \\ -i\gamma_2\omega + \left(\frac{\partial \xi_5^T}{\partial \mu}\right) i\Omega \cdot \mathbf{k} & -i\gamma_3\omega + \left(\frac{\partial \xi_5^T}{\partial \mu_5}\right) i\Omega \cdot \mathbf{k} \end{bmatrix}. \quad (5.34)$$

The superscript T denote that we are studying the vector and axial charge currents at $T = \text{Const.}$. Two characteristics associated with the above matrix are the following CVWs

$$\omega_{1,2} = -\frac{\mathcal{A}_6 \pm \sqrt{\mathcal{A}_6^2 - \mathcal{E}\mathcal{A}_7}}{\mathcal{E}} \Omega \cdot \mathbf{k} \quad (5.35)$$

where \mathcal{A}_6 and \mathcal{A}_7 are two complicated expression constructed out of anomaly coefficients together with the susceptibility matrix elements. Instead of writing \mathcal{A}_6 and \mathcal{A}_7 generally, we focus just on the heavy ion fluid case wherein $\mu_5 = 0$. In this case $\beta_2 = \gamma_3 = \chi$ and $\beta_3 = \gamma_2 = 0$ and so the CVW may be given by the following dispersion relations

$$\omega_{1,2} = \pm \frac{\mathcal{C}}{2\pi^2 \chi} \sqrt{1 - \frac{\mu n}{w} - \frac{\pi^2 n T^2}{3 \mu w}} \Omega \cdot \mathbf{k} \quad (5.36)$$

which clearly differs from that of found in [32], namely (5.33). Let us clarify the reason for this difference. While our computations in this paper have been all done in the Landau frame, the authors of [32] have found their results in the laboratory frame. Associatively, the transport coefficients in laboratory frame, namely those used in equations (1) and (2) in [32], are the same as the transport coefficients in Landau frame (2.6), up to terms without explicit n or n_5 dependence. As a result, the equation (5.36) has to give the velocity of CVW in the laboratory frame, when removing the terms with n dependence. It can be simply seen that under the latter consideration, (5.36) coincides with (5.33).

Although we could have reproduced the results related to the laboratory frame from those related to the Landau frame, a question remains unresolved; why are the hydrodynamic modes non-frame-invariant?¹⁵ In an upcoming paper [39], we will fully discuss on this fact that the hydrodynamic modes are frame-invariant in a free fluid. By free fluid we mean a fluid with all possible perturbations free to propagate around the equilibrium in the system. However the fluid considered in the current subsection is actually a forced fluid; the perturbations of the temperature and momentum have been forced to be turned off, allowing just the axial and vector currents to fluctuate. In a forced fluid like what has been studied above, the dispersion relations of the hydrodynamic modes depend on the choice of the frame.

5.4.3 Chiral Heat Wave (CHW)

The CHW was firstly found in [37] in the laboratory frame. In this subsection we will derive CHW in the Landau frame¹⁶.

Recently, Chernodub has shown that in a “forced” rotating chiral fluid, a new type of grapples excitations propagates; the so-called Chiral Heat Wave (CHW) is a coherent propagation of the energy, axial and vector currents [37]. It has been shown that when the local momentum perturbations in the fluid are turned off, the pure CHW propagates with the following velocity at $\mu = \mu_5 = 0$:

$$v_{CHW} = \frac{\Omega}{6} \sqrt{\frac{T^3}{c_v \chi}}. \quad (5.37)$$

with $c_v = \partial \epsilon / \partial T$. It is important to note that the above result have been found in the laboratory frame. As mentioned at the end of the previous subsection, when the fluid is forced, the hydro modes are not frame-invariant. Moreover, in the case of CHW we need to use the energy current in the laboratory frame which includes

¹⁵Let us emphasis that by the frame here we mean a fluid frame in which we have considered a specific definition for the velocity of the fluid, so this has nothing with the notion of Lorentz frame.

¹⁶This subsection has been added in the second version of the paper, following the discussions between Maxim Chernodub and the authors.

anomalous terms as the following

$$\mathbf{j}_E^{Lab} = T_{Lab}^{0i} \propto \sigma_{\epsilon,B} \mathbf{B} + \sigma_{\epsilon,\omega} \mathbf{\Omega}. \quad (5.38)$$

The above anomalous contributions did not enter in our computations in the Landau frame at all. So instead of CHW, we found the CMVW in the Landau frame. In what follows we will show that when momentum perturbations are forced to be turned off, CHW may propagate in the fluid.

In this case, the 6-dimensional state-space reduces to a 3-dimensional space, including the states characterized by a vector chemical potential, an axial chemical potential and a temperature. The matrix (5.4) so simplifies to

$$M = \begin{bmatrix} -i\alpha_1\omega & -i\alpha_2\omega & 0 \\ -i\beta_1\omega & -i\chi\omega & ik\Omega \left(\frac{\mu}{2\pi^2} - \frac{n}{w} \left(\frac{T^2}{6} + \frac{\mu^2}{2\pi^2} \right) \right) \\ i\frac{T}{6}k\Omega & i\frac{\mu}{2\pi^2}k\Omega & -i\chi\omega \end{bmatrix}. \quad (5.39)$$

The corresponding excitations are

$$\omega_1 = 0 \quad (5.40)$$

$$\omega_{2,3} = \pm \frac{\Omega}{6} \sqrt{\frac{T^3}{(\epsilon/T)\chi}} \frac{n}{\sqrt{(1+c_s^2)(w\chi - n^2/c_s^2)}} k. \quad (5.41)$$

The last two mods above are CHWs in the Landau frame. As it can be clearly seen, here and in contrast to the result in laboratory frame, CHW needs the presence of a finite vector charge density to propagate.

Now let us repeat the above computations in the laboratory frame. In this frame the matrix M takes the following form:

$$M = \begin{bmatrix} -i\alpha_1\omega & 0 & i\frac{T^2}{6}k\Omega \\ 0 & -i\chi\omega & 0 \\ i\frac{T}{6}k\Omega & 0 & -i\chi\omega \end{bmatrix}. \quad (5.42)$$

The corresponding excitations are

$$\omega_1 = 0 \quad (5.43)$$

$$\omega_{2,3} = \pm \frac{\Omega}{6} \sqrt{\frac{T^3}{c_v\chi}} k. \quad (5.44)$$

Obviously, the last two excitations are exactly CHWs found in [37]. This difference between the velocity of CHW in Landau and the Laboratory frame is another manifestation of the non-frame-invariance of the hydro modes in a forced fluid which was discussed in previous subsection.

6 Conclusion and Outlook

As a main part in this paper we have found the hydrodynamic excitations in a fluid carrying both vector and axial charges. Neglecting the dissipative effects, none of these excitations are entropy producing; they are either adiabatic or anomalous waves in the fluid. In the latter case, the chiral transport may be observed in the fluid when fluid is coupled to an external magnetic field or is rotating around an axis.

In this paper, we have considered a general case in which the fluid is in presence of a constant magnetic field \mathbf{B} and simultaneously is rotating with a constant vorticity $\mathbf{\Omega}$. It has been shown that the full spectrum of the collective excitations constitute six modes in general; two of them are the coherent perturbations of the scalar currents, namely J_E^μ, J^μ, J_5^μ , while another four modes are made out of perturbations of all six scalar and vector currents.

The scalar modes are the mixture of well-known CMW and CVW. Indeed when $\mathbf{\Omega} = 0$, our results give the generalization of the well-known CMW associated with the zero density [28]. Similarly, we have obtained the generalization of the well-known CVW associated with zero axial density, when we take the $\mathbf{B} = 0$ in our result [32].

In the scalar-vector sector, we find four mixed Sound-Alfvén-Coriolis modes which are all dispersive in general. When $\mathbf{\Omega} = 0$, these modes become the mixed Sound-Alfvén modes. However the Alfvén part here is something different from the standard Alfvén wave in magnetohydrodynamics; the origin of this wave in our work is purely hydrodynamical. Furthermore, at finite axial charge density, we have found the momentum perturbations in the fluid may propagate in the direction of the magnetic field; this is analogous to the chiral Alfvén wave firstly found in [34, 35].

All results in this paper have been found in presence of a non-dynamical magnetic field. It would be interesting to investigate how a dynamical magnetic field coupled to the flowing matter may affect on the nature of the excitations¹⁷. To this end, one has to find the full spectrum of the chiral magnetohydrodynamics. We leave this issue for the future studies.

In another direction, it would be of more interest to find the spectrum of the hydrodynamic excitations propagating on top of the expanding quark gluon plasma¹⁸. Recently, the authors of [42] have studied the linear fluctuations around a Bjorken flow analytically although, neither they coupled the fluid to the magnetic field nor the chiral transport was considered in their work. It would be phenomenologically important to extend the subject of [42] to the chiral QCD case.

Apart from the quark gluon plasma, our results found in this paper may be applied to other phenomena in physics as well. A different place to explore is indeed the neutrino matter at the core of the supernovae star wherein, a gas of noninteracting

¹⁷See [40, 41, 43] for recent studies.

¹⁸The first attempt in this way, including CME however mostly numerically, was made in [36].

fermions is flowing [44]. It would be interesting to see how the velocities of the hydro waves change with the density there. We leave further study on the issue to our future work [45].

Acknowledgements

We would like to thank Prof. Mohsen Alishahiha for encouragements and supporting the Larak-Particle-Pheno group. We would also like to thank M. Mohammadi Najafabadi for reading the paper thoroughly and giving useful comments. We thank A. Akhavan. N.A. Would like to thank Prof. H. Arfaei for illuminating discussions on gravitational anomaly. A.D would like to thank P. V. Buividovich and S. N. Valgushev for discussion. The work of A.D was supported by the S. Kowalevskaja award from the Alexander von Humboldt Foundation. We would like to thank Maxim Chernodub for discussion.

References

- [1] A. Vilenkin, “Equilibrium Parity Violating Current In A Magnetic Field,” *Phys. Rev. D* **22**, 3080 (1980).
- [2] A. Vilenkin, “Macroscopic Parity Violating Effects: Neutrino Fluxes From Rotating Black Holes And In Rotating Thermal Radiation,” *Phys. Rev. D* **20**, 1807 (1979).
- [3] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, “Fluid dynamics of R-charged black holes,” *JHEP* **0901**, 055 (2009).
- [4] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, and P. Surowka, “Hydrodynamics from charged black branes,” *JHEP* **1101**, 094 (2011).
- [5] D. T. Son and N. Yamamoto, “Berry Curvature, Triangle Anomalies, and the Chiral Magnetic Effect in Fermi Liquids,” *Phys. Rev. Lett.* **109**, 181602 (2012); “Kinetic theory with Berry curvature from quantum field theories,” *Phys. Rev. D* **87**, 085016 (2013).
- [6] M. A. Stephanov and Y. Yin, “Chiral Kinetic Theory,” *Phys. Rev. Lett.* **109**, 162001 (2012).
- [7] J. -W. Chen, S. Pu, Q. Wang, and X. -N. Wang, “Berry curvature and 4-dimensional monopole in relativistic chiral kinetic equation,” *Phys. Rev. Lett.* **110**, 262301 (2013).
- [8] P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya and M. I. Polikarpov, “Numerical evidence of chiral magnetic effect in lattice gauge theory,” *Phys. Rev. D* **80**, 054503 (2009) [arXiv:0907.0494 [hep-lat]].

- [9] P. V. Buividovich, M. N. Chernodub, D. E. Kharzeev, T. Kalaydzhyan, E. V. Luschevskaya and M. I. Polikarpov, “Magnetic-Field-Induced insulator-conductor transition in SU(2) quenched lattice gauge theory,” *Phys. Rev. Lett.* **105**, 132001 (2010) [arXiv:1003.2180 [hep-lat]].
- [10] M. Pühr and P. V. Buividovich, arXiv:1611.07263 [hep-lat].
- [11] P. V. Buividovich and S. N. Valgushev, *PoS LATTICE* **2016**, 253 (2016) [arXiv:1611.05294 [hep-lat]].
- [12] D. T. Son and P. Surowka, *Phys. Rev. Lett.* **103** (2009) 191601 [arXiv:0906.5044 [hep-th]].
- [13] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*. Pergamon, 1987.
- [14] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and A. Yarom, *JHEP* **1405** (2014) 147 [arXiv:1105.3733 [hep-th]].
- [15] Y. Neiman and Y. Oz, *JHEP* **1103**, 023 (2011).
- [16] J.H. Gao, Z.T. Liang, S. Pu, Q. Wang, and X.N. Wang “Chiral Anomaly and Local Polarization Effect from the Quantum Kinetic Approach,” *Phys. Rev. Lett.* **109**, 232301 (2012)
- [17] K. Landsteiner, E. Megias, and F. Pena-Benitez, “Gravitational Anomaly and Transport,” *Phys. Rev. Lett.* **107**, 021601 (2011); *Lect. Notes Phys.* **871**, 433 (2013).
- [18] A. V. Sadofyev and M. V. Isachenkov, *Phys. Lett. B* **697**, 404 (2011) [arXiv:1010.1550 [hep-th]].
- [19] P. Kovtun, “Lectures on hydrodynamic fluctuations in relativistic theories,” *J. Phys. A* **45**, 473001 (2012) [arXiv:1205.5040 [hep-th]].
- [20] N. Abbasi and A. Davody, “Dissipative Charged Fluid in a Magnetic Field,” *Phys. Lett. B* **756**, 161 (2016) [arXiv:1508.06879 [hep-th]].
- [21] D. E. Kharzeev and H. U. Yee, “Anomalies and time reversal invariance in relativistic hydrodynamics: the second order and higher dimensional formulations,” *Phys. Rev. D* **84** (2011) 045025 [arXiv:1105.6360 [hep-th]].
- [22] S. Sen and N. Yamamoto, “Chiral Shock Waves,” arXiv:1609.07030 [hep-th].
- [23] N. Yamamoto, “Chiral transport of neutrinos in supernovae,” arXiv:1611.06076 [astro-ph.HE].
- [24] M. Kaminski, C. F. Uhlemann, M. Bleicher and J. Schaffner-Bielich, “Anomalous hydrodynamics kicks neutron stars,” *Phys. Lett. B* **760**, 170 (2016) doi:10.1016/j.physletb.2016.06.054 [arXiv:1410.3833 [nucl-th]].
- [25] M. Giovannini and M. E. Shaposhnikov, “Primordial hypermagnetic fields and

- triangle anomaly,” Phys. Rev. D **57**, 2186 (1998) doi:10.1103/PhysRevD.57.2186 [hep-ph/9710234].
- [26] K. Landsteiner, “Anomalous transport of Weyl fermions in Weyl semimetals,” Phys. Rev. B **89**, no. 7, 075124 (2014) doi:10.1103/PhysRevB.89.075124 [arXiv:1306.4932 [hep-th]].
 - [27] K. Landsteiner, “Notes on Anomaly Induced Transport,” arXiv:1610.04413 [hep-th].
 - [28] D. E. Kharzeev and H. U. Yee, “Chiral Magnetic Wave,” Phys. Rev. D **83** (2011) 085007 [arXiv:1012.6026 [hep-th]].
 - [29] Y. Burnier, D. E. Kharzeev, J. Liao and H. U. Yee, “Chiral magnetic wave at finite baryon density and the electric quadrupole moment of quark-gluon plasma in heavy ion collisions,” Phys. Rev. Lett. **107**, 052303 (2011) [arXiv:1103.1307 [hep-ph]].
 - [30] L. Adamczyk *et al.* [STAR Collaboration], “Observation of charge asymmetry dependence of pion elliptic flow and the possible chiral magnetic wave in heavy-ion collisions,” Phys. Rev. Lett. **114**, no. 25, 252302 (2015) [arXiv:1504.02175 [nucl-ex]].
 - [31] R. Belmont [ALICE Collaboration], “Charge-dependent anisotropic flow studies and the search for the Chiral Magnetic Wave in ALICE,” Nucl. Phys. A **931**, 981 (2014) [arXiv:1408.1043 [nucl-ex]].
 - [32] Y. Jiang, X. G. Huang and J. Liao, “Chiral vortical wave and induced flavor charge transport in a rotating quark-gluon plasma,” Phys. Rev. D **92**, no. 7, 071501 (2015) [arXiv:1504.03201 [hep-ph]].
 - [33] Dalton.D. Schnack, Lectures in Magnetohydrodynamics. Springer-Verlag. Berlin. Heidelberg. 2009.
 - [34] N. Yamamoto, “Chiral Alfvén Wave in Anomalous Hydrodynamics,” Phys. Rev. Lett. **115**, no. 14, 141601 (2015) [arXiv:1505.05444 [hep-th]].
 - [35] N. Abbasi, A. Davody, K. Hejazi and Z. Rezaei, “Hydrodynamic Waves in an Anomalous Charged Fluid,” Phys. Lett. B **762**, 23 (2016) [arXiv:1509.08878 [hep-th]].
 - [36] S. F. Taghavi and U. A. Wiedemann, “Chiral magnetic wave in an expanding QCD fluid,” Phys. Rev. C **91** (2015) no.2, 024902 [arXiv:1310.0193 [hep-ph]].
 - [37] M. N. Chernodub, “Chiral Heat Wave and mixing of Magnetic, Vortical and Heat waves in chiral media,” JHEP **1601**, 100 (2016) [arXiv:1509.01245 [hep-th]].
 - [38] D. Frenklakh, Phys. Rev. D **94**, no. 11, 116010 (2016) doi:10.1103/PhysRevD.94.116010 [arXiv:1603.08971 [hep-th]].
 - [39] N. Abbasi, D. Allahbakhshi, A. Davody, S.F. Taghavi, “To Appear”

- [40] M. Giovannini, “Anomalous Magnetohydrodynamics,” *Phys. Rev. D* **88**, 063536 (2013).
- [41] A. K. Pandey, “A study on the collective behavior of chiral plasma using first and second order conformal hydrodynamics,” arXiv:1609.01848 [hep-ph].
- [42] Y. Akamatsu, A. Mazeliauskas and D. Teaney, “A kinetic regime of hydrodynamic fluctuations and long time tails for a Bjorken expansion,” arXiv:1606.07742 [nucl-th].
- [43] N. Sadooghi and S. M. A. Tabatabaee, “The effect of magnetization and electric polarization on the anomalous transport coefficients of a chiral fluid,” arXiv:1612.02212 [hep-th].
- [44] N. Yamamoto, “Chiral transport of neutrinos in supernovae: Neutrino-induced fluid helicity and helical plasma instability,” *Phys. Rev. D* **93**, no. 6, 065017 (2016) [arXiv:1511.00933 [astro-ph.HE]].
- [45] N. Abbasi, D. Allahbakhshi, A. Davody, S.F. Taghavi, “To Appear”